ECONOMICS

2nd Semester

Paper – C4T

(Mathematical Methods in Economics – II)

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Function of Several variables

(Partial derivatives)

1.Q: A production function is given by $Q = \frac{1}{3}K^3L^3$, where Q is the level of output and K and L are capital and labour inputs. Obtain the marginal productivities of capital and labour.

Ans: Marginal productivities of capital (MP_K) and labour (MP_L) can be obtained by partial differentiation of Q with respect to K and L respectively.

Thus $MP_K \equiv \frac{\delta Q}{\delta K} = K^2L^3$ and $MP_L \equiv \frac{\delta Q}{\delta L} = K^3L^2$

2. Differentiate partially $z = 3x^2 + 3y^2x$ with respect to x twice.

Ans:- First differentiation gives us the first order partial derivative with respect to x which is

$$\frac{\delta z}{\delta x} \equiv f_x = 6x + 3y^2$$

Now, we can repeat the process of differentiation with respect to x to get the second order partial derivative which is

$$\frac{\delta^2 z}{\delta x^2} \equiv f_{xx} = 6$$

3. Find f_{11} and f_{22} for the function $y = (x_1 + 2x_2)^2$

Ans: To find out f_{11} and f_{22}, first we have to derive f_1 and f_2, which are in fact the first order partial derivatives of y. Now

$$f_1 = 2(x_1 + 2x_2) = 2x_1 + 4x_2 \text{ and}$$

$$f_2 = 2(x_1 + 2x_2) \cdot 2 = 4x_1 + 8x_2$$

Repeating the process of differentiation, we get

$$\frac{\delta f_1}{\delta x_1} = f_1 = 2 \quad \text{and} \quad \frac{\delta f_2}{\delta x_2} = f_{22} = 8$$

4. Given $z = x+y$, Could you find f_{xx} and f_{yy} for this function $z = f(x, y)$?

Ans:- Here $f_x = 1$ and $f_y = 1$. Since both f_x and f_y are constants, their further differentiation will be zero. Hence $f_{xx} = 0$ and $f_{yy} = 0$

Consider the function $y = f(x_1, x_2)$. Here the number of choice variable is two. Hence we get two first order partial derivatives.

$$\frac{\delta f}{\delta x_1} \text{ or } f_1 \text{ and } \frac{\delta f}{\delta x_2} \text{ or } f_2$$

and four second order partial derivatives

$$\frac{\delta f_1}{\delta x_1} \text{ or } f_{11} \text{ and } \frac{\delta f_1}{\delta x_2} \text{ or } f_{12}, \frac{\delta f_2}{\delta x_2} \text{ or } f_{22} \text{ and } \frac{\delta f_2}{\delta x_1} \text{ or } f_{21}$$

Among these four second order partial derivatives two, f_{12} and f_{21}, are called cross second order partial derivatives or mixed partial derivatives while the other two, f_{11} and f_{22}, are called direct second order partial derivatives.

5. Given $y = \log(x_1^2 + x_2^2)$, find the second order partial derivatives.

Ans: Here, $f_1 = \frac{1}{x_1^2+x_2^2}\cdot 2x_1 = \frac{2x_1}{x_1^2+x_2^2}$

$$f_2 = \frac{1}{x_1^2+x_2^2}\cdot 2x_2 = \frac{2x_2}{x_1^2+x_2^2}$$

So, $$f_{11} = \frac{(x_1^2+x_2^2)\times 2 - 2x_1(2x_1)}{(x_1^2+x_2^2)} = \frac{2(x_2^2-x_1^2)}{(x_1^2+x_2^2)^2}$$

and $$f_{22} = \frac{(x_1^2+x_2^2)\times 2 - 2x_2(2x_2)}{(x_1^2+x_2^2)^2} = \frac{2(x_1^2-x_2^2)}{(x_1^2+x_2^2)^2}$$

Similarly, $$\frac{(x_1^2+x_2^2)\times \frac{\delta}{\delta x_2}(2x_1) - 2x_1\cdot \frac{\delta}{\delta x_2}(x_1^2+x_2^2)}{(x_1^2+x_2^2)^2}$$

$$= \frac{2x_1(2x_2)}{(x_1^2+x_2^2)^2} = -\frac{4x_1x_2}{(x_1^2+x_2^2)^2}$$

and $$f_{21} = \frac{(x_1^2+x_2^2)\cdot \frac{\delta}{\delta x_1}(2x_2) - 2x_2\cdot \frac{\delta}{\delta x_1}(x_1^2+x_2^2)}{(x_1^2+x_2^2)^2}$$

$$= \frac{2x_2(2x_1)}{(x_1^2+x_2^2)^2} = -\frac{4x_1x_2}{(x_1^2+x_2^2)^2}$$

6. Given $z = e^{2x+2y}$, show that $f_{xy} - f_{yx} = 0$

Ans:- Here, $f_x = e^{2x+2y} \cdot \frac{\delta}{\delta x}(2x+2y) = e^{2x+2y} \cdot (2) = 2e^{2x+2y}$

and $f_{xy} = 2e^{2x+2y} \cdot \frac{\delta}{\delta y}(2x+2y) = 2e^{2x+2y} \cdot (2) = 4e^{2x+2y}$

Again, $f_y = e^{2x+2y} \cdot \frac{\delta}{\delta y}(2x+2y) = e^{2x+2y} \cdot (2) = 2e^{2x+2y}$

and $f_{yx} = 2e^{2x+2y} \cdot \frac{\delta}{\delta x}(2x+2y) = 2e^{2x+2y} \cdot (2) = 4e^{2x+2y}$

Hence, $f_{xy} - f_{yx} = 0$

Total Differential

1. Given $Q = 4x_1 + 3x_2$, $x_1 = t^3 + t^2 + 1$, $x_2 = t^3 - t^2 - t$, find $\frac{dQ}{dt}$.

Ans:- Here, $\frac{dQ}{dt} = f_1 \cdot \frac{dx_1}{dt} + f_2 \cdot \frac{dx_2}{dt}$ where $f_1 = \frac{\delta Q}{\delta x_1}$

and $f_2 = \frac{\delta Q}{\delta x_2}$

Now, $f_1 = \frac{\delta Q}{\delta x_1} = 4$ and $f_2 = \frac{\delta Q}{\delta x_2} = 3$

Further, $\frac{dx_1}{dt} = 3t^2 + 2t$ and $\frac{dx_2}{dt} = 3t^2 - 2t - 1$

Then $\frac{dQ}{dt} = 4(3t^2 + 2t) + 3(3t^2 - 2t - 1)$

$= 21t^2 + 2t - 3$

[Alternative way, $Q = 4(t^3 + t^2 + 1) + 3(t^3 - t^2 - t)$

$= 4t^3 + 4t^2 + 4 + 3t^3 - 3t^2 - 3t$

$= 7t^3 + t^2 - 3t + 4$

$\therefore \frac{dQ}{dt} = (7 \times 3)t^2 + 2t - 3$]

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Total Differential

Consider a function $y = f(x_1, x_2)$. By total differential of this function we measure the total change of y due to a change in both x_1 and x_2, when x_1 and x_2 are assumed to be independent of each other. Then if dy is denoted to be the total change in y when x_1 is changed by dx_1 and x_2 is changed by dx_2 simultaneously. We can write

$$dy = f_1 dx_1 + f_2 dx_2$$

Here, dy is called the total differential of the function $y = f(x_1, x_2)$.

2. Find the total differential of $y = 3x_1 + 5x_2$

Ans: From this function we get,

$$\frac{\delta y}{\delta x_1} \equiv f_1 = 3 \quad \text{and} \quad \frac{\delta y}{\delta x_2} \equiv f_2 = 5$$

Then $dy = = 3dx_1 + 5dx_2$

3. We are given $U = x_1^2 + x_2^2$, such that $x_1 = 2x_2^2 + 2$

Obtain $\frac{du}{dx_2}$.

Ans: [Let us now assume that x_1 is a function x_2. Suppose we have the function $y = f(x_1, x_2)$ where $x_1 = g(x_2)$.

Thus, we get,

$$\frac{dy}{dx_2} = f_1 \frac{dx_1}{dx_2} + f_2 \;]$$

In the above Question No. 3, we have

$$\frac{du}{dx_2} = f_1 \frac{dx_1}{dx_2} + f_2 \, , \quad \text{since } x_1 = g(x_2)$$

In our present problem

$$f_1 = 2x_1 \, , \quad f_2 = 2x_2 \text{ and } \frac{dx_1}{dx_2} = 4x_2$$

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$$f_1 = 2x_1, \quad f_2 = 2x_2 \text{ and } \frac{dx_1}{dx_2} = 4x_2$$

Thus

$$\frac{du}{dx_2} = 2x_1 \cdot 4x_2 + 2x_2$$

$$= 2x_2 \cdot 2x_1 \cdot 2 + 2x_2$$

$$= 2x_2(4x_1 + 1)$$

$$= 2x_2\{4(2x_2^2 + 2) + 1\}$$

$$\cancel{= 2x_2(8x_2^2 + 8 + 1)}$$

$$= 8x_2(2x_2^2 + 2) + 2x_2$$

Alternative way, By substitution we get,

$$u = (2x_2^2 + 2)^2 + x_2^2$$

Now, differentiating w.r.t. to x_2 we get,

$$\frac{du}{dx_2} = 2(2x_2^2 + 2) \cdot 4x_2 + 2x_2$$

$$= 8x_2(2x_2^2 + 2) + 2x_2$$

Total Derivative

The concept of total derivative is associated with multivariate functions. By total derivative we measure the rate of change of the dependent variable owing to any change in a variable on which it depends, when none of the variables is assumed to be constant as in the case of partial differentiation. Let a bivariate function be given by $y = f(x_1, x_2)$.

Such that $x_1 = g(t)$ and $x_2 = h(t)$

Let us suppose that we are interested to know the rate of change of y owing to a change in t, ie to know $\frac{dy}{dt}$. Then we can write,

$$\frac{dy}{dt} = \frac{\delta y}{\delta x_1} \cdot \frac{dx_1}{dt} + \frac{\delta y}{\delta x_2} \cdot \frac{dx_2}{dt}$$

written alternatively,

$$\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt}$$

Here, $\frac{dy}{dt}$ is called the total derivative of y with respect to t.

[example:- See, in the above question no:-1]

Explicit and implicit functions

Functional relation between two variables exists when there exists a dependency relation between the two. But sometimes it is seen that the two variables are so related that we cannot clearly indicate their order of dependency, ie we fail to indicate which one is the dependent and which one is the independent variable. Such a function is called an implicit function. For example,

$2x - 3y + 1 = 0$ is an implicit function in x and y

From this function we can get

(i) $2x = 3y - 1$

or $x = \frac{3}{2}y - \frac{1}{2}$ is an explicit function of y.

(ii) Again, we can also get,

$3y = 2x + 1$

or $y = \frac{2}{3}x + \frac{1}{3}$ is an explicit function of x.

Thus, from an implicit function we can get two explicit functions. These two explicit functions are said to be inverse to each other.

Exercise

1. Obtain the differential, dy:

(i) $y = \frac{1}{5} + x^3$ (ii) $y = (6-x)/(2-5x^2)$

(iii) $y = (5-x^2)^7$

2. Find $\frac{dy}{dx}$ in the following cases:

(i) $y = -9x^{-4}$ (ii) $y = (4x^2-3)(2x^5)$

(iii) $y = e^x x^n$ (iv) $y = \log(ax^2+bx+c)$

(v) $e^{\sqrt{(1+x)}} - e^{\sqrt{(x-1)}}$ (vi) $y = e^{7x}$

(vii) $y = x^2 e^x$

3. If $x = 2t+3$, $y = t^2-1$, obtain $\frac{dy}{dx}$.

4. obtain the explicit functions corresponding to the following functions:

(i) $x^3 - 9y^2 = 0$ (ii) $x^2 - 3y + 5 = 0$

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