4th semester

Summations ($\Sigma \to$ sigma):

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

(i) $$\sum_{i=1}^{n} (x_i \pm y_i) = \sum_{i=1}^{n} x_i \pm \sum_{i=1}^{n} y_i$$

Proof:- $\Sigma (x_i \pm y_i) = [(x_1 + x_2 + \cdots + x_n) \pm (y_1 + y_2 + \cdots + y_n)]$

$= \Sigma x_i \pm \Sigma y_i$

(ii) $$\sum_{i=1}^{n} K x_i = K \sum_{i=1}^{n} x_i$$

Proof:- $\Sigma K x_i = K x_1 + K x_2 + \cdots K x_n$

$= K(x_1 + x_2 + \cdots + x_n)$

$= K \Sigma x_i$

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(iii) $\sum_{i=1}^{n} K = nK$

Proof:- We may write $\sum K = \sum (K X_i)$, where all the x's are equal to 1. Thus we have

$$\sum K = \sum K X_i = K X_1 + K X_2 + \cdots + K X_n$$

But $X_1 = X_2 = \cdots = X_n = 1$. Therefore

$$\sum_{i=1}^{n} K = K(1) + K(1) + \cdots K(1) = (K + K + \cdots K) = nK$$

: The algebra of expected values :

1. $E(X \pm Y) = E(X) \pm E(Y)$

Proof:- $E(X+Y) = \sum_i \sum_j (X_i + Y_j) f(X_i, Y_j) = \sum_i \sum_j X_i f(X_i, Y_j) + \sum_i \sum_j Y_j f(X_i, Y_j)$

$$= \sum_i X_i \sum_j f(X_i, Y_j) + \sum_j Y_j \sum f(X_i, Y_j)$$

$$= \sum_i X_i f(X_i) + \sum_j Y_j f(Y_j) = E(X) + E(Y)$$

2. $E(K) = K$

Proof: $E(K) = \sum_i K f(K) = K \sum f(K) = K$

given $\sum f(K)$ = Sum of probabilities = 1

3. $E(KX_i) = KE(X)$

Proof: $E(KX_i) = \sum (KX_i) f(X_i) = K \sum X_i f(X_i) = KE(X)$.

4. $var(X+Y) = var(X) + var(Y) + 2\,cov(XY)$

Proof: $var(X+Y) = E[(X+Y) - E(X+Y)]^2$

$$= E\{[X - E(X)] + [Y - E(Y)]\}^2$$

$$= E[X - E(X)]^2 + E[Y - E(Y)]^2 + 2E\{[X - E(X)][Y - E(Y)]\}$$

$$= var(X) + var(Y) + 2\,cov(XY)$$

$$= var(X) + var(Y) + 2\,cov(XY).$$

5. $$E(XY) = E(X)E(Y)$$

Proof: $$E(XY) = \sum_i \sum_j x_i y_j f(x_i, y_j)$$

$$= \sum_i \sum_j x_i y_j f(x_i) f(y_j) \quad [\because f(x_i, y_j) = f(x_i) f(y_j)]$$

$$= [\sum x_i f(x_i)] [\sum_j y_j f(y_j)]$$

$$= E(X) E(Y)$$

6. $cov(XY) = 0$, for independent variables

Proof: $$cov(XY) = E\{[X - E(X)][Y - E(Y)]\}$$

$$= E\{XY - YE(X) - XE(Y) + E(X)E(Y)\}.$$

$$= E(XY) - E(X)E(Y) \quad$$ [By rule 5, for independ variables $E(XY) = E(X)E($

$$= E(X)E(Y) - E(X)E(Y) = 0$$

7. $Var(aX+b) = a^2 var(X)$

Proof: $var(aX+b) = E\{(aX+b) - E(aX+b)\}^2$

$$= E\{(aX+b) - [aE(X)+b]\}^2$$

$$= E[aX+b-a\mu-b]^2$$

$$= a^2 E(X-\mu)^2 = a^2 var(X)$$

8. $E\left(\sum_{i=1}^{n} X_i Y_i\right) = \sum_{i=1}^{n} [E(X_i Y_i)]$

Proof:- $E\left(\sum_{i=1}^{n} X_i Y_i\right) = E(X_1 Y_1 + X_2 Y_2 + \cdots + X_n Y_n)$

$$= E(X_1 Y_1) + E(X_2 Y_2) + \cdots + E(X_n Y_n)$$

$$= \sum_{i=1}^{n} [E(X_i Y_i)]$$

9. $E\left(\sum_{i=1}^{n} X_i Y_i\right)^2 = E\left[\sum_{i=1}^{n} X_i^2 Y_i^2 + 2\sum_{i\neq j} X_i X_j Y_i Y_j\right]$

Proof:- $E\left(\sum_{i=1}^{n} X_i Y_i\right)^2 = E[X_1 Y_1 + X_2 Y_2 + \cdots + X_n Y_n]^2$

$$= E(X_1^2 Y_1^2 + X_2^2 Y_2^2 + \cdots + X_n^2 Y_n^2 + 2X_1 X_2 Y_1 Y_2 + 2X_1 X_3 Y_1 Y_3 + \cdots]$$

$$= E\left[\sum_{i=1}^{n} X_i^2 Y_i^2 + 2\sum_{i\neq j}^{n} X_i X_j Y_i Y_j\right]$$

2) Show how the total variation in dependent variable of the linear regression model can be decomposed into two parts: explained variation and unexplained variation.

Ans: Economic theory assumes that the functional relationship between variables are exact under the ceteris paribus clause. For example, the demand function $D = b_0 + b_1 P$ postulated by economic theory implies that the quantity of a particular commodity is a linear function of its price alone, 'other things remaining equal'; that is, the price-quantity relationship holds provided that all other factors not appearing explicitly in the function (for example tastes, income, other prices) remain unchanged.

In econometrics we may read the true relationship connecting the variables as follows. Y is connected with X by a linear relationship, ceteris paribus. If factors other than X remain unchanged then changes in Y would be fully explained by changes in X. However, other factors do not remain equal; hence we introduce u into the function to account for the changes in other variables not included in it explicitly. We may now look at the final form of our equation

$$Y_i = b_0 + b_1 X_i + u_i$$

Y_i $(i = 1, 2, \ldots, n)$ can be expressed in terms of two components, one component due to X_i and a second component due to the influences included in the random term u_i.

$$\underbrace{Y_i}_{\text{[Total variation]}} = \underbrace{b_0 + b_1 X_i}_{\text{[Explained variation]}} + \underbrace{u_i}_{\text{[unexplained variation]}}$$

So, we can say that the true relationship which can be expressed into two parts;- Explained variation and unexplained variation.

Gauss- Markov least-squares theorem.

The least squares estimates are BLU (best, linear, unbiased) provided that the random term u satisfies some general assumptions, namely that the u has zero mean and constant variance. This proposition, together with the set of conditions under which it is true, is known as Gauss- Markov least-squares theorem.

The OLS estimators possess three properties: they are linear, unbiased and have the smallest variance.

1. : **Linearity**:

The least-squares estimates $\hat{b}_0$ and $\hat{b}_1$ are linear functions of the observed sample values Y_i

(a) We established that

$$\hat{b}_1 = \sum \frac{x_i}{\sum x_i^2} Y_i = \sum K_i Y_i \text{ , where } K_i = \frac{x_i}{\sum x_i^2}$$

But, the values of X are a set of fixed numbers in all samples from sample to sample. Hence, the K_i's are fixed constants from sample to sample. We may write,

$$\begin{aligned} \hat{b}_1 &= \sum K_i Y_i \\ &= K_1 Y_1 + K_2 Y_2 + \cdots + K_n Y_n \\ &= f(Y) \end{aligned}$$

The estimate $\hat{b}_1$ is a linear function of the Y's, a linear combination of the values of the dependent variable.

(b) Now, we have,

$$\hat{b}_0 = \Sigma\left[\frac{1}{n} - \bar{X}K_i\right]Y_i$$, where $\bar{X}$ and K_i are fixed constants from sample to sample. Thus $\hat{b}_0$ depends only on the values of Y, that is, $\hat{b}_0$ is a linear function of the sample values of Y.

2. Unbiasedness:

An estimator is unbiased if its expected value is equal to the true parameter, that is if $E(\hat{b}) = b$.

We established that,

$$\hat{b}_1 = \Sigma K_i Y_i$$

By substituting the value of $Y_i = b_0 + b_1 X_i + u_i$ then we may write,

$$\hat{b}_1 = \Sigma K_i (b_0 + b_1 X_i + u_i)$$

$$= b_0 \Sigma K_i + b_1 \Sigma K_i X_i + \Sigma K_i u_i$$

But $\Sigma K_i = 0$ and $\Sigma K_i X_i = 1$

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(i) $\sum K_i = \frac{\sum x_i}{\sum x_i^2} = \frac{\sum (X_i - \bar{X})}{\sum x_i^2} = \frac{0}{\sum x_i^2} = 0$

(ii) $\sum K_i X_i = \frac{\sum x_i X_i}{\sum x_i^2} = \frac{\sum (X_i - \bar{X}) X_i}{\sum x_i^2} = \frac{\sum X_i^2 - \bar{X} \sum X_i}{\sum x_i^2} = 1$

$[\because \sum X_i^2 - \bar{X} \sum X_i = \sum x_i^2]$

$$\therefore \hat{b}_1 = b_1 + \sum K_i u_i = b_1 + \frac{\sum x_i u_i}{\sum x_i^2}$$

$$\therefore E(\hat{b}_1) = E(b_1) + E\left(\frac{\sum x_i u_i}{\sum x_i^2}\right)$$

$$= b_1 + \frac{\sum x_i E(u_i)}{\sum x_i^2}$$

$$= b_1$$

Again, $E(\hat{b}_0) = b_0$

We have established

$$\hat{b}_0 = \bar{Y} - \hat{b}_1 \bar{X}$$

Substituting $\hat{b}_1 = \sum K_i Y_i$, we obtain

$$\hat{b}_0 = \bar{Y} - \bar{X} \sum K_i Y_i$$

$$= \frac{\sum Y_i}{n} - \bar{X} \sum K_i Y_i$$

$$= \sum \left[\frac{1}{n} - \bar{X} K_i\right] Y_i$$

Now, $E(\hat{b}_0) = \sum \left[\frac{1}{n} - \bar{X} K_i\right] E(Y_i)$

$$= \sum \left[\frac{1}{n} - \bar{X} K_i\right] (b_0 + b_1 X_i) \quad [\because E(Y_i) = b_0 + b_1 X_i]$$

$$= \sum \left[\frac{b_0}{n} - \bar{X} K_i b_0 + \frac{b_1 X_i}{n} - \bar{X} K_i b_1 X_i\right]$$

$$= \frac{n \times b_0}{n} - \bar{X} b_0 \sum K_i + b_1 \bar{X} - \bar{X} b_1 \sum K_i X_i$$

$$= b_0 - 0 + b_1 \bar{X} - b_1 \bar{X} \quad [\because \sum K_i = 0, \sum K_i X_i = 1]$$

$$= b_0$$

Minimum-variance estimator (Or best estimator):

An estimate is best when it has the smallest variance as compared with any other estimate obtained from other econometric methods. Symbolically $\hat{b}$ is best if

$$E\left[\hat{b} - E(\hat{b})\right]^2 < E\left(\tilde{b} - E(\tilde{b})\right]^2$$

$$\text{or,} \quad Var(\hat{b}) < var(\tilde{b})$$

where, $\tilde{b}$ is any other (not necessarily unbiased) estimate of the true parameter b.

Proof:- we have established,

$$Var(\hat{b}_1) = \sigma_u^2 \frac{1}{\Sigma x_i^2}$$

We want to prove that,

$$Var(\hat{b}_1) < Var(\tilde{b}_1)$$

Firstly,
let us assume, $\tilde{b}_1 = \Sigma c_i Y_i$, where $c_i = K_i + d_i$ and K_i being the weights for the OLS estimates, and d_i an arbitrary set of weights similar (but not the same) to the K_i's. Substituting $b_0 + b_1 X_i + u_i$ for Y_i we obtain

$$\tilde{b}_1 = \Sigma c_i (b_0 + b_1 X_i + u_i) = \Sigma (b_0 c_i + b_1 c_i X_i + c_i u_i)$$

Secondly, The new estimate $\tilde{b}_1$ is also assumed to be an unbiased estimator of the true b_1, that is that is $E(\tilde{b}_1) = b_1$, Taking expected values

$$E(\tilde{b}_1) = E\left[b_0 \Sigma c_i + b_1 \Sigma c_i X_i + \Sigma (c_i u_i)\right]$$

Now, $E(\tilde{b}_1) = b_1$ if, and only if,

$$\Sigma c_i = 0, \quad \Sigma c_i X_i = 1 \text{ and } \Sigma c_i u_i = 0$$

But $\Sigma c_i = 0$ implies $\Sigma d_i = 0$, because

$$\Sigma c_i = \Sigma (k_i + d_i) = \Sigma k_i + \Sigma d_i = 0$$

$$\text{So, } \Sigma d_i = 0, \left[\Sigma k_i = 0\right]$$

Similarly, $\Sigma c_i X_i = 1$ requires $\Sigma d_i X_i = 0$, since $\Sigma c_i X_i = \Sigma k_i X_i + \Sigma d_i X_i$ and given that $\Sigma k_i X_i = 1$

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In summary, Since we are defining $\tilde{b}_1$ to be a linear unbiased estimate of b_1, with weights $c_i = K_i + d_i$, it follows that,

$$\Sigma c_i = 0, \quad \Sigma d_i = 0, \quad \Sigma c_i X_i = 1, \quad \Sigma d_i X_i = 0$$

<u>Thirdly</u>, the variance of the new estimator $\tilde{b}_1$ is

$$var(\tilde{b}_1) = var(\hat{b}_1) + \sigma_u^2 \Sigma d_i^2$$

Proof:- $\hat{b}_1 = \Sigma K_i Y_i$

$$var(\hat{b}_1) = var(\Sigma K_i Y_i) = \Sigma var(K_i Y_i) = \Sigma [K_i^2 var(Y)]$$
$$= \Sigma K_i^2 \sigma_u^2 = \sigma_u^2 \Sigma K_i^2$$

Now, $\tilde{b} = \Sigma c_i Y_i$

$$var(\tilde{b}) = var(\Sigma c_i Y_i) = \Sigma c_i^2 var(Y_i)$$

given that the c_i's are constant weights, independent of the Y_i's.

But, $var(Y_i) = \sigma_u^2$

Therefore, $var(\tilde{b}_1) = \sigma_u^2 \Sigma c_i^2$

Now, $\Sigma c_i^2 = \Sigma (K_i + d_i)^2 = \Sigma K_i^2 + \Sigma d_i^2 + 2\Sigma K_i d_i$

$$= \Sigma K_i^2 + \Sigma d_i^2$$

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Now, $\tilde{b} = \Sigma c_i Y_i$

$$var(\tilde{b}) = var(\Sigma c_i Y_i) = \Sigma c_i^2 var(Y_i)$$

given that the c_i's are constant weights, independent of the Y_i's.

But, $var(Y_i) = \sigma_u^2$

Therefore, $var(\tilde{b}_1) = \sigma_u^2 \Sigma c_i^2$

Now, $\Sigma c_i^2 = \Sigma (k_i + d_i)^2 = \Sigma k_i^2 + \Sigma d_i^2 + 2\Sigma k_i d_i$

$$= \Sigma k_i^2 + \Sigma d_i^2$$

given that, $\Sigma k_i d_i = \Sigma \frac{x_i}{\Sigma x_i^2} d_i$

$$= \frac{\Sigma (X_i - \bar{X}) d_i}{\Sigma x_i^2}$$

$$= \frac{\Sigma d_i X_i - \bar{X}\Sigma d_i}{\Sigma x_i^2} = 0 \quad \left[\because \Sigma d_i X_i = 0 \text{ and } \Sigma d_i = 0\right]$$

Now, $var(\tilde{b}_1) = \sigma_u^2 (\Sigma k_i^2 + \Sigma d_i^2)$

$$= \sigma_u^2 \Sigma k_i^2 + \sigma_u^2 \Sigma d_i^2$$

But, $\sigma_u^2 \Sigma k_i^2 = var(\hat{b}_1)$; therefor

$$var(\tilde{b}_1) = var(\hat{b}_1) + \sigma_u^2 \Sigma d_i^2$$

$\therefore$ $var(\tilde{b}_1) > var(\hat{b}_1)$

Proof :- 2 $\quad var(\tilde{b}_0) > var(\hat{b}_0)$

we have, $\hat{b}_0 = \Sigma \left[\frac{1}{n} - \bar{X} k_i\right] Y_i$

and $\tilde{b}_0 = \Sigma \left[\frac{1}{n} - \bar{X} c_i\right] Y_i$

We substitute for $Y_i = b_0 + b_1 X_i + u_i$ in $\tilde{b}_0$

$$\tilde{b}_0 = b_0\left[1 - \bar{X}\Sigma c_i\right] + b_1\left[\bar{X} - \bar{X}\Sigma c_i X_i\right] + \Sigma\left[\frac{1}{n} - \bar{X} c_i\right] u_i$$

$$E(\tilde{b}_0) = b_0\left[1 - \bar{X}E(\Sigma c_i)\right] + b_1\left[\bar{X} - \bar{X}E(\Sigma c_i X_i)\right] + E\left[\Sigma\left(\frac{1}{n} - \bar{X}c_i\right)u_i\right]$$

Now, $E(\tilde{b}_0) = b_0$ if, and only if

$\Sigma c_i = 0$, $\Sigma c_i X_i = 1$ and $\Sigma c_i u_i = 0$.

These conditions imply, $\Sigma d_i = 0$ and $\Sigma d_i X_i = 0$

Now, $var(\tilde{b}_0) = E(\tilde{b}_0 - b_0)^2$

$$= \sigma_u^2 \Sigma\left[\frac{1}{n} - \bar{X}c_i\right]^2$$

$$= \sigma_u^2 \Sigma\left[\frac{1}{n^2} + \bar{X}^2 c_i^2 - 2\frac{1}{n}\bar{X}c_i\right]$$

$$= \sigma_u^2\left[\frac{n}{n^2} + \bar{X}^2\Sigma c_i^2 - 2\bar{X}\frac{1}{n}\Sigma c_i\right]$$

$$= \sigma_u^2\left[\frac{1}{n} + \bar{X}^2\Sigma c_i^2 - \frac{2}{n}\bar{X}c_i\right]$$

given that, $\Sigma c_i = 0$ and $\Sigma c_i^2 = \Sigma k_i^2 + \Sigma d_i^2$, we have

$$var(\tilde{b}_0) = \sigma_u^2\left[\frac{1}{n} + \bar{X}^2(\Sigma k_i^2 + \Sigma d_i^2)\right]$$

$$= \sigma_u^2\left[\frac{1}{n} + \frac{\bar{X}^2}{\Sigma x^2}\right] + \left[\sigma_u^2\bar{X}^2\Sigma d_i^2\right]$$

$\therefore$ $var(\tilde{b}_0) = var(\hat{b}_0) + \sigma_u^2\left[\bar{X}^2\Sigma d_i^2\right]$

$\left[\because \sigma_u^2\left[\frac{1}{n} + \frac{\bar{X}^2}{\Sigma x^2}\right] = var(\hat{b}_0)\right]$

$\therefore$ $var(\tilde{b}_0) > var(\hat{b}_0)$

Reverse (or inverse) regression method :

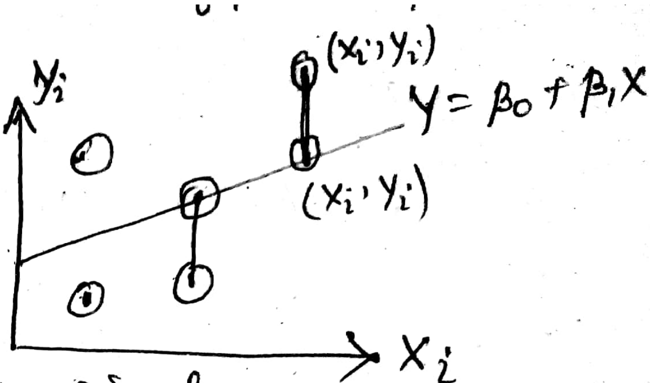
Consider a simple linear regression model

$$y = \beta_0 + \beta_1 x + u$$

where y is dependent variable and x is independent or explanatory variable. The term β_0 and β_1 ~~and~~ are the parameters of the model. The parameter β_0 is as intercept term and the parameter β_1 is ~~term~~ as slope parameter. These parameters are usually called as regression coefficient. u is the unobservable error component and that is normally distributed, i.e. zero mean and constant variance σ^2, i.e. $u \sim N(0, \sigma^2)$

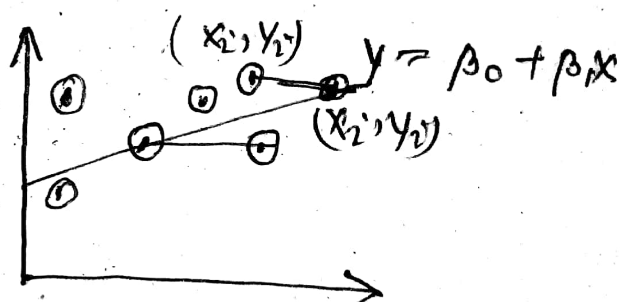
Various methods of estimation can be used to determine the estimates of the parameters. Among them, the methods of least squares and maximum likelihood are the popular methods of estimation.

The method of least squares estimates the parameters β_0 and β_1 by minimizing the sum of squares of difference between the observations and the line in the scatter diagram. When the vertical difference between the observations and the line in the scatter diagram is considered and its sum of squares is minimized to obtain the estimates of β_0 and β_1, the method is known as direct regression.



Direct regression method

Alternatively, the sum of squares of difference between the observations and the line in horizontal direction in the scatter diagram can be minimized to obtain the estimates of β_0 and β_1. This is known as revers (or inverse) regression method.



Reverse regression method

PRATIMA BHAKTA

ECONOMICS

4th Semester

Paper - C10T

(Introductory Econometrics)