MCQ
Solutions

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To my beloved Daughters

Samadrita & Somdatta

Preface

With the remarkable advancement in various branches of science, engineering and technology, today more than ever before, the study of differential equations has become essential. For, to have an exhaustive understanding of subjects like physics, mathematical biology, chemical science, mechanics, fluid dynamics, head transfer, aerodynamics, electricity, waves and electromagnetic, the knowledge of finding solution to differential equations is absolutely necessary. These differential equations may be ordinary or partial. Finding and interpreting their solutions are at the heart of applied mathematics. A thorough introduction to differential equations is therefore a necessary part of the education of any applied mathematician, and this book is aimed at building up skills in this area.

This book on ordinary / partial differential equations is the outcome of a series of lectures delivered by me, over several years, to the undergraduate or postgraduate students of Mathematics at various institution. My principal objective of the book is to present the material in such a way that would immediately make sense to a beginning student. In this respect, the book is written to acquaint the reader in a logical order with various well-known mathematical techniques in differential equations. Besides, interesting examples solving JAM / GATE / NET / IAS / SSC questions are provided in almost every chapter which strongly stimulate and help the students for their preparation of those examinations from graduate level.

Organization of the book
The book has been organized in a logical order and the topics are discussed in a systematic manner. It has comprising 19 chapters altogether. In the chapter 1, the fundamental concept of differential equations including autonomous/ non-autonomous and linear / non-linear differential equations has been explained. The order and degree of the ordinary differential equations (ODEs) and partial differential equations(PDEs) are also mentioned. The chapter 2 are concerned the first order and first degree ODEs. It is also written in a progressive manner, with the aim of developing a deeper understanding of ordinary differential equations, including conditions for the existence and uniqueness of solutions. In chapter **??** the first order and higher degree ODEs are illustrated with sufficient examples. The chapter **??** is concerned with the higher order and first degree ODEs. Several methods, like method of undetermined coefficients, variation of parameters and Cauchy-Euler equations are also introduced in this chapter. In chapter **??**, second order initial value problems, boundary value problems and Eigenvalue problems with Sturm-Liouville problems are expressed with proper examples. Simultaneous linear differential equations are studied in chapter **??**. It is also written in a progressive manner with the aim of developing some alternative methods. In chapter **??**, the equilibria, stability

and phase plots of linear / nonlinear differential equations are also illustrated by including numerical solutions and graphs produced using Mathematica version 9 in a progressive manner. The geometric and physical application of ODEs are illustrated in chapter **??**. The chapter **??** is presented the Total (Pfaffian) Differential Equations. In chapter 3, numerical solutions of differential equations are added with proper examples. Further, I discuss Fourier transform in chapter **??**, Laplace transformation in chapter **??**, Inverse Laplace transformation in chapter **??**. Moreover, series solution techniques of ODEs are presented with Frobenius method in chapter **??**, Legendre function and Rodrigue formula in Chapter **??**, Chebyshev functions in chapter **??**, Bessel functions in chapter **??** and more special functions for Hypergeometric, Hermite and Laguerre in chapter **??** in detail.

Besides, the partial differential equations are presented in chapter **??**. In the said chapter, the classification of linear, second order partial differential equations emphasizing the reasons why the canonical examples of elliptic, parabolic and hyperbolic equations, namely Laplace's equation, the diffusion equation and the wave equation have the properties that they do has been discussed. Also all chapters are concerned with sufficient examples. In addition, there is also a set of exercises at the end of each chapter to reinforce the skills of the students.

By reading this book, I hope that the readers will appreciate and be well prepared to use the wonderful subject of differential equations.

Aim and Scope
When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results of the modelling process is a system of ordinary or partial differential equations. Finding and interpreting the solutions of these differential equations is therefore a central part of applied mathematics, Physics and a thorough understanding of differential equations is essential for any applied mathematician and physicist. The aim of this book is to develop the required skills on the part of the reader. The book will thus appeal to undergraduates/postgraduates in Mathematics, but would also be of use to physicists and engineers. There are many worked examples based on interesting real-world problems. A large selection of examples / exercises including JAM/NET/GATE questions is provided to strongly stimulate and help the students for their preparation of those examinations from graduate level. The coverage is broad, ranging from basic ODE , PDE to second order ODE's including Bifurcation theory, Sturm-Liouville theory, Fourier Transformation, Laplace Transformation and existence and uniqueness theory, through to techniques for nonlinear differential equations including stability methods. Therefore, it may be used in research organization or scientific lab.

Significant features of the book

- A complete course of differential Equations
- Perfect for self-study and class room
- Useful for beginners as well as experts
- More than 500 worked out examples
- Large number of exercises
- More than 600 multiple choice questions with answers
- Suitable for GATE, NET, JAM, JEST, IAS, SSC examinations.

Use of software
The software package Latex version 5.3 was used to write the book. Mathematica version 9 was used to obtain the phase curve, eigenvalue for checking the stability of a dynamical system and solve the different equations. Lingo version 8 was also some time used to obtain the numerical results. All these packages were able to solve problems in material requirements planning and project management techniques easily.

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This book is the outcome of a series of lectures and research experience carried out by me over several years. However it would not be possible to incorporate or framing the entire book without the help of many academicians. As such, I am indebted to many of my teachers and students. Especially I would like to thank Dr. Swapan Kumar Misra, Principal, Mugberia Gangadhar Mahavidyalaya for his generous support in writing this book.

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I shall feel great to receive constructive criticisms through email for the improvement of the book from the experts as well as the learners.

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Contents

Chapter 1

Metric Space

1.1 Introduction

1.2 Multiple Choice Questions(MCQ)

1. Let d_1, d_2 and d_3 be metrics on a set X with at least two elements. Which of the following is NOT a metric on X? **Gate(MA): 2014**
(a) Min $\{d_1, 2\}$ (b) Max $\{d_2, 2\}$ (c) $\frac{d_3}{1+d_3}$ (d) $\frac{d_1+d_2+d_3}{3}$.
Ans. (b)

2. Let d_1, d_2 be the following metrics on $\mathbb{R}^n$ where $d_1(x, y) = \sum\limits_1^n |x_i - y_i|$, $d_2(x, y) = \left(\sum\limits_1^n |x_i - y_i|^2\right)^{\frac{1}{2}}$.
Then decide which of the following is a metric on $\mathbb{R}^n$. **NET(MS)(Dec.): 2016**
(a) $d(x, y) = \frac{d_1(x,y)+d_2(x,y)}{1+d_1(x,y)+d_2(x,y)}$
(b) $d(x, y) = d_1(x, y) - d_2(x, y)$
(c) $d(x, y) = d_1(x, y) + d_2(x, y)$
(d) $d(x, y) = e^{\pi} d_1(x, y) + e^{-\pi} d_2(x, y)$
Ans. (a), (c) and (d).

3. Consider the metric $d_2(f, g) = \left(\int\limits_a^b |f(t) - g(t)|^2\right)^{\frac{1}{2}}$ and $d_\infty(f, g) = \sup\limits_{t\in[a,b]} |f(t) - g(t)|$ on the space $X = C[a, b]$ of all real values of continuous functions on $[a, b]$. Then which of the following is TRUE? **Gate(MA): 2009**
(a) Both (X, d_2) and $((X, d_\infty)$ are complete.
(b) (X, d_2) is complete but $((X, d_\infty)$ is not complete.
(b) (X, d_∞) is complete but $((X, d_2)$ is not complete.
(d) Both (X, d_2) and $((X, d_\infty)$ are not complete.
Ans. (a).

4. Which of the following is / are true? **NET(MS)(Jun): 2016**
(a) $(0, 1)$ with the usual topology admits a metric which is complete
(b) $(0, 1)$ with the usual topology admits a metric which is not complete
(c) $[0, 1]$ with the usual topology admits a metric which is not complete
(d) $[0, 1]$ with the usual topology admits a metric which is complete.
Ans. (b) and (c).

5. Consider the smallest topology τ on $\mathbb{C}$ in which all the singleton sets are closed. Pick each correct statement from below: **NET(MS)(Jun): 2016**
(a) $(\mathbb{C}, \tau)$ is Housdorff. (b) $(\mathbb{C}, \tau)$ is compact.
(c) $(\mathbb{C}, \tau)$ is connected. (d) $\mathbb{Z}$ is dence in $(\mathbb{C}, \tau)$.
Ans. (a), (b) and (c).

6. Let A the following subset of $\mathbb{R}^2$: $\{(x, y) : (x + 1)^2 + y^2 \leq 1\} \bigcup \{(x, y) : y = x \sin \frac{1}{x}, x > 0\}$. Then **NET(MS)(Dec.): 2016**
(a) A is connected (b) A is compact
(c) A is path connected (d) A is bounded
Ans. (b) and (d).

7. Let $(\mathbb{R}, \tau)$ be a topological space with the confinite topology. Every infinite subset of $\mathbb{R}$ is
(a) Compact but not connected (b) Both compact and connected **Gate(MA): 2016**
(c) Not compact but connected (d) Neither compact nor connected
Ans. (b).

8. $f : [0, 1] \to [0, 1]$ is called shrinking map if $|f(x) - f(y) < |x - y|$ for all $x, y \in [0, 1]$ and a contraction if there exist a $\alpha < 1$ such that $|f(x) - f(y) < \alpha|x - y|$ for all $x, y \in [0, 1]$. Which of the following statements is TRUE for the function $f(x) = x - \frac{x^2}{2}$?. **Gate(MA): 2016**
(a) f is both a shrinking map and a contraction
(b) f is a shrinking map but NOT a contraction
(c) f is NOT a shrinking map but a contraction
(d) f is Neither a shrinking map NOT a contraction
Ans. (b) and (d).

9. Let d_1 and d_2 denoted the usual metric and discrete metric on $\mathbb{R}$ respectively.
Let $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_2)$ be denoted by $f(x) = x,\ x \in \mathbb{R}$. Then **Gate(MA): 2015**
(a) f is continuous but f^{-1} is NOT continuous
(b) f^{-1} is continuous but f is NOT continuous
(c) both f and f^{-1} are continuous
(d) neither f nor f^{-1} is continuous
Ans. (b)

10. If $I : \left(l^1 : \|\cdot\|_2\right) \to \left(l^2 : \|\cdot\|_1\right)$ is the identity map, then **Gate(MA): 2009**
(a) I is continuous but I^{-1} is NOT continuous
(b) I^{-1} is continuous but I is NOT continuous
(c) both I and I^{-1} are continuous
(d) neither I nor I^{-1} is continuous
Ans. (c)

11. Let X be a non-empty set. Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two topologies on X such that $\mathfrak{I}_1$ is strictly contained in $\mathfrak{I}_2$. If $I : (X, \mathfrak{I}_1) \to (X, \mathfrak{I}_2)$ is the identity map, then **Gate(MA): 2008**
(a) both I and I^{-1} are continuous
(b) neither I nor I^{-1} is continuous
(c) I is continuous but I^{-1} is NOT continuous
(d) I^{-1} is continuous but I is NOT continuous
Ans. (c)
Hint. Since $I(\mathfrak{I}_1) = \mathfrak{I}_1 \subset \mathfrak{I}_2$ but $I(\mathfrak{I}_2) = \mathfrak{I}_2 \subset \mathfrak{I}_1$. Hence the result.

12. Which of the following subsets of $\mathbb{R}^2$ is NOT compact? **Gate(MA): 2013**
(a) $\{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1,\ y = \sin x\}$
(b) $\{(x, y) \in \mathbb{R}^2 : -1 \le y \le 1,\ y = x^8 - x^3 - 1\}$
(c) $\{(x, y) \in \mathbb{R}^2 : y = 0,\ \sin(e^{-x}) = 0\}$
(d) $\{(x, y) \in \mathbb{R}^2 : x > 0, y = \sin \frac{1}{x}\} \bigcap \{(x, y) \in \mathbb{R}^2 : x > 0, y = \frac{1}{x}\}$
Ans. (c)

13. Which of the following sets are compact ?.
(a) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1$ in the Euclidean topology. **NET(MS)(Dec.): 2015**
(b) $\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1^2 + z_2^2 + z_3^2 = 1$ in the Euclidean topology.
(c) $\prod_1^n A_n$ with product topology, where $A_n = \{0, 1\}$ has discrete topology for $n = 1, 2, 3, \cdots$.
(d) $\{z \in \mathbb{C} : |Rez \le a|$ in the Euclidean topology for some fixed positive real number a.
Ans. (a) and (c).

14. Let G_1 and G_2 be two subsets of $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function. Then
(a) $f^{-1}(G_1 \bigcup G_2) = f^{-1}(G_1) \bigcup f^{-1}(G_2)$
(b) $f^{-1}(G_1^c) = (f^{-1}(G_1))^c$ **NET(MS)(Dec.): 2015**
(c) $f^{-1}(G_1 \bigcap G_2) = f^{-1}(G_1) \bigcap f^{-1}(G_2)$
(d) If G_1 is open and G_2 is closed then, $G_1 + G_2 = \{x + y : x \in G_1,\ y \in G_2$ is neither open nor closed.
Ans. (a) and (b).

15. Let f be a bounded function on $\mathbb{R}$ and $a \in \mathbb{R}$. For $\delta > 0$, $\omega(a, \delta) = \sup |f(x) - f(a)|$, $x \in (a - \delta, a + \delta)$. Then
(a) $\omega(a, \delta_1) \le \omega(a, \delta_2)$ if $\delta_1 \le \delta_2$
(b) $\lim\limits_{\delta \to 0^+} \omega(a, \delta) = 0$ for all $a \in \mathbb{R}$. **NET(MS)(Jun): 2015**
(c) $\lim\limits_{\delta \to 0^+} \omega(a, \delta)$ need not exist.
(d) $\lim\limits_{\delta \to 0^+} \omega(a, \delta) = 0$ if and only if f is continuous at a.
Ans. (a) and (d).

16. Consider the set $\mathbb{Z}$ of integers with the topology τ in which a subset is closed if and only if it is empty or $\mathbb{Z}$ or finite. Which of the following statement is true? **NET(MS)(Jun): 2015**
(a) τ is the subspace topology induced from the usual topology on $\mathbb{R}$
(b) $\mathbb{Z}$ is compact in the topology τ
(c) $\mathbb{Z}$ is Hausdorff in the topology τ
(d) Every infinite subset of $\mathbb{Z}$ is dence in the topology τ
Ans. (b) and (d).

17. The subspace $P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2 + 1\}$ is
(a) Compact and connected (b) Compact but not connected **Gate(MA): 2011**
(c) Not compact but connected (d) Neither compact nor connected
Ans. (c).

18. For which subspace $X \subseteq \mathbb{R}$ with the usual topology and with $\{0, 1\} \subseteq X$ will a continuous function $f : X \to \{0, 1\}$ satisfying $f(0) = 0$ and $f(1) = 1$ exist ? **Gate(MA): 2011**
(a) $X = [0, 1]$ (b)$X = [-1, 1]$ (c) $X = \mathbb{R}$ (d) $[0, 1] \not\subseteq X$
Ans. (d).

19. Suppose X be a finite set of more than fives elements. Which of the following is TRUE?
(a) There is a topology on X which is T_3
(b) There is a topology on X which is T_2 but not T_3. **Gate(MA): 2011**
(c) There is a topology on X which is T_1 but not T_2.
(d) There is no topology on X which is T_1
Ans. (a).

20. The set $X = \mathbb{R}$ with the metric $d(x, y) = \frac{|x-y|}{1+|x-y|}$ is
(a) bounded but not compact (b) bounded but not complete **Gate(MA): 2010**
(c) Complete but not bounded (d) Compact but not complete
Ans. (b) and (d).

21. Let $X = N$ be equipped with the topology generated by the basis consisting of sets $A_n = \{n, n+1, n+2, \cdots\} : n \in N$. Then X is
(a) Compact and connected (b) Hausdorff and connected **Gate(MA): 2010**
(c) Hausdorff and compact (d) Neither compact not connected
Ans. (d).

22. Let $X = N \times Q$ with the subspace topology of the usual topology on $\mathbb{R}^2$ and $P = \{(n, \frac{1}{n}) : n \in N\}$. In the space X
(a) P is closed but not open (b) P is open but not closed **Gate(MA): 2010**
(c) P is both open and closed (d) P is neither open nor closed.
Ans. (d).

23. Let $X = N \times Q$ with the subspace topology of the usual topology on $\mathbb{R}^2$ and $P = \{(n, \frac{1}{n}) : n \in N\}$. The boundary of P in X is **Gate(MA): 2010**
(a) an empty set (b) a singleton set (c) P (d) X.
Ans. (d).

24. In a topological space, which of the following statements is NOT always true ?
(A) Union of any finite family of compact sets is compact. **Gate(MA): 2012**
(B) Union of any family of closed sets is closed.
(C) Union of any family of connected sets having a non empty intersection is connected.
(D) Union of any family of dense subsets is dense.
Ans. (d).

25. Consider the following statements:
P: The family of subsets $\left\{A_n = \left(-\frac{1}{n}, \frac{1}{n}\right), n = 1, 2, \cdots\right\}$ satisfies the finite intersection property. **Gate(MA): 2012**
Q: On an infinite set X , a metric $d : X \times X \to R$ is defined as $d(x, y) = 0, x = y$ and $d(x, y) = 1, x \neq y$.
The metric space (X, d) is compact.
R: In a Frechet (T_1) topological space, every finite set is closed.
S: If $f : R \to X$ is continuous, where R is given the usual topology and (X, τ) is a Hausdorff (T_2) space, then f is a one-one function.
Which of the above statements are correct?
(A) P and R (B) P and S (C) R and S (D) Q and S.
Ans. (c).

26. Let $X = \{a, b, c\}$ and let $\zeta = \left\{\phi, \{a\}, \{b\}, \{a, b\}, X\right\}$ be a topology defined on X. Then which of the following statements are TRUE? **Gate(MA): 2012**
$P : (X, \zeta)$ is a Hausdorff space. $Q : (X, \zeta)$ is a regular space.
$R : (X, \zeta)$ is a normal space. $S : (X, \zeta)$ is a connected space.
(A) P and Q (B) Q and R (C) R and S (D) P and S.
Ans. (b).

Chapter 2

Complex Analysis

Magnitude and Angle of a complex number: Let $z = x + iy$ be a complex number. Then magnitude of z is given by $r = |z| = \sqrt{x^2 + y^2}$ and argument of z is given by $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. The principal argument of the multi-valued argument is between $-\pi$ and $+\pi$ i.e., $-\pi < \theta \leq \pi$.

Polar form of a complex number: If $x = r\cos\theta$, $y = r\sin\theta$, then $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

DeMoivre's Theorem: $z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$.
Analytic functions: A function $\omega = f(z)$ is said to be analytic at a point z_0 if $f(z)$ is differentiable not only at z_0 but also at every point of some neighbourhood of z_0. A function that is analytic throughout the whole complex plane is called an entire function.

Necessary and sufficient condition for an analytic function: If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u, v satisfy the equations. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ Provided the four partial derivatives u_x, u_y, v_x, v_y exist.

Cauchy Riemann equations: If $f(z) = u(x, y) + iv(x, y)$ be an analytic function, then

1. **Cartesian Form:** $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$
2. **Polar Form:** $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Milne Thomson Theorem: This method is used for finding analytic function $f(z)$ when either real or imaginary part is given.

(i) When u is given

$$\frac{\partial u}{\partial x} = \varphi_1(x, y)$$

$$\frac{\partial u}{\partial y} = \varphi_2(x, y)$$

Then $f(z) = \int \left\{\varphi_1(z, 0) - i\varphi_1(z, 0)_2\right\} dz + C$

(ii) When v is given

$$\frac{\partial v}{\partial x}=\psi_2(x,y)$$

$$\frac{\partial v}{\partial y}=\psi_1\,(x,y)$$

Then $f(z)=\int\left\{\psi_1(z,0)+i\psi_2(z,0)_2\right\}dz+C$

L'Hopital's Rule: For two functions g(z) and h(z) that are differentiable at z_0 andIf g(z_0)and h(z_0) are both 0 and If h′(z_0) is NOT equal to 0 then $\lim\limits_{z\to z_0}\frac{g(z)}{h(z)}=\frac{g'(z_0)}{h'(z_0)}$. Extension to this rule: if g(z), h(z), and their first n derivatives vanish at z_0, then $\lim\limits_{z\to z_0}\frac{g(z)}{h(z)}=\frac{g^{(n+1)}(z_0)}{h^{(n+1)}(z_0)}$.

Harmonic Functions: Any function satisfying Laplace's equation is said to be harmonic. Wherever a function is analytic, its real and imaginary parts are harmonic. The real and imaginary parts of harmonic functions are call conjugates of one another, i.e., $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ and $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$

Taylor's Theorem: A function $f(z)$ which is analytic at all points with in a circle with center at z_0 and of radius R can be represented uniquely as a convergent power series given by $f(z)=a_n(z-z_0)^n$, where$a_n=\frac{f^n(z)}{n!}$.

Important Results

$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$	$\sin i\theta=i\sinh\theta$, $\cos i\theta=i\cosh\theta$
$\cosh\theta=\frac{e^{\theta}+e^{-\theta}}{2}$, $\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}$	$\sinh z=\frac{e^{z}-e^{-z}}{2}$, $\cosh z=\frac{e^{z}+e^{-z}}{2}$
$\sin z=\sin x\cosh y+i\cos x\sinh y$	$\sin iz=i\sinh z$, $\cos iz=i\cosh z$
$\cos z=\cos x\cosh y+i\sin x\sinh y$	$\log z=Log\,r+i\theta\;for\;(r\neq 0)$
$\log z=Log\,\lvert z\rvert+i\arg z\;for\;(z\neq 0)$	$e^{\log(z+i2\pi)}=z$
$\cosh^2 z-\sinh^2 z=1$	$e^z=e^{x+iy}=e^x(\cos y+i\sin y)$

Cauchy's integral theorem: If *f(z)* is analytic and single valued inside and on a simple closed contour C, then$\int_C f(z)dz=0$.
Linville Theorem: If *f(z) is continuous on a contour C of length l and if M be the upper bound of* $\left|f(z)\right|$ *on C, then* $\left|\oint_c f(z)\,dz\right|\leq M\,l$.
Morera's theorem: If a function $f(z)$ is continuous in a domain D and such that of $\oint_G f(z)\,dz=0$, for every simple contour G in D , then f(z) is analytic in D.
Cauchy's integral formula: If *f(z)* is analytic within and on a closed contour C, and if a is any point within C, then $f^{(n)}(a)=\frac{n!}{2\pi i}\int_C\frac{f(z)dz}{(z-a)^{n+1}}$.
Taylor's Theorem: If f(*z)* is analytic within a circle C with its center z = a and radius R, then at every point z inside C, then$f(z)=\sum\limits_{n=0}^{\infty}a_n(z-a)^n$, where $a_n=\frac{f^n(a)}{n!}$.
Laurent' series: If *f(z)* is analytic in the closed ring bounded by two concentric circles C and C′ of centre a and radius R and R′, (R′ < R). If z is any point of the annulus, then$f(z)=\sum\limits_{n=0}^{\infty}a_n(z-a)^n$

$+\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ where $a_n = \frac{n!}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{(z-a)^{n+1}}$ and $b_n = \frac{n!}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{(z-a)^{-n+1}}$.

Cauchy Residue theorem: If *f(z)* is analytic within and on a closed contour C, except at a finite number of poles $z_1, z_2, z_3, \cdots, z_n$ within C, then $\int_C f(z)dz = 2\pi i \sum_{r=1}^{n} Res\ (z = z_r) = 2\pi i \times (sum\ of\ residue)$.

(a) For simple pole

(i) Res $(z = a) = \lim_{z\to a}(z-a)f(z)$.

(ii) $Res(z=a) = \frac{\varphi(a)}{\psi\acute{}(a)}$ if $f(z) = \frac{\varphi(z)}{\psi(z)}$.

(b) For multiple pole

(i) $Res(z=a) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]$

(ii) $Res\,(z=a) = coefficient\ \ of\ \frac{1}{t}$ where $t = z-a$.

2.1 Multiple Choice Questions

1. If real part of an analytic faction $f(z) = u + iv$ is $u = x^2 - y^2$, then the analytic function is
   (a) $f(z) = iz^2 + c$ (b) $f(z) = -iz^2 + c$ (c) $f(z) = z + c$ (d) $f(z) = z^2 + c$.
   Ans. (d) Use Milne Thomson Formula, we have $f(z) = \int \{\varphi_1(z,0) - i\varphi_1(z,0)\}dz + C$.

2. If imaginary part of an analytic faction $f(z)$ is $v = e^x(xsiny + ycosy)$, then the analytic function is
   (a) $f(z) = ize^2 + c$ (b) $f(z) = -ize^2 + c$ (c) $f(z) = ze^z + c$ (d) $f(z) = z^2 + c$.
   Ans. (c) Use Milne Thomson Formula, we have $f(z) = \int \left\{\psi_1(z,0) + i\psi_2(z,0)_2\right\}dz + C$

3. If $\sin z = \sum_{n=0}^{\infty} a_n\left(z - \frac{\pi}{4}\right)^n$, then a_6 equals to
   (a) 0 (b) $\frac{1}{720\sqrt{2}}$ (c) $\frac{1}{720}$ (d) $-\frac{1}{720\sqrt{2}}$
   Ans. (d) Use Taylors Theorem

4. The value of $\int_{|z|=2}\left(\frac{e^{3z}}{z-1}\right)dz$
   (a) $2\pi i$ (b) $2\pi ie^3$ (c) 0 (d) $2\pi e$
   Ans. (b) Cauchy Residue theorem.

5. For the positively oriented unit circle, $\oint_{|z|=1}\frac{2Re(z)}{z+2}dz =$ GATE(MA) : 2004
   (a) 0 (b) $\pi\, i$ (c) $2\pi\, i$ (d) $4\pi\, i$
   Ans. (a)

6. The residues of a complex function $f(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles are
   (a) $\frac{1}{2}, -\frac{1}{2}, and\ 0$ (b) $\frac{1}{2}, -\frac{1}{2}, and -1$ (c) $1, -\frac{1}{2}, and -\frac{3}{2}$ (d) $\frac{1}{2}, -\frac{1}{2}, and\frac{3}{2}$.
   Ans. (c)

7. If $f(z) = \frac{z}{8-z^3}, z = x + iy$. Then Residue of $f(z)$ at $z = 2$ is
   (a) $-\frac{1}{8}$ (b) $\frac{1}{8}$ (c) $-\frac{1}{6}$ (d) $\frac{1}{6}$ GATE(MA): 2011
   Ans. (d)

8. If a function $f(z)$ is continuous in region D and if $\int_D f(z)\,dz = 0$, taken around any simple closed contour in D. Then $f(z)$ is

(a) Non-Analytic (b) Analytic (c) may or may not be Analytic (d) none of these
Ans. (b) Morera's Theorem.

9. If a function $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in region D. Then u, v are satisfied by the following equations
(a) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ (b) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}$
(c) $\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ (d) $\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$
Ans. (a)

10. Let γ be the curve $r = 2 + 4\cos\theta\ 0 < \theta < \pi)$ if $I_1 = \int_\gamma \frac{dz}{z-1}$ and $I_2 = \int_\gamma \frac{dz}{z-2}$. Then
(a)$I_1 = 2I_2$ (b)$I_1 = I_2$ (c)$2I_1 = I_2$ (d)$I_1 = 0, \quad I_2 \neq 0$
Ans. (b)

11. The value $\oint_C (z-10)^{10} dz$ is equals to (where C is the contour $|z - 10| = 50$)
(a) $2\pi i$ (b)$-2\pi i$ (c) $2\pi i \times 10^9$ (d) 0.
Ans. (d)

12. The value $\oint_C \frac{e^{-2z}}{(z+1)^3} dz$ is equals to (where C is the contour $|z| = 2$)
(a) $2\pi i$ (b)$-4\pi i$ (c) $4\pi i$ (d) 0.
Ans. (d)

13. The value $\oint_{|z|=2} \tan z\, dz$ is equals to
(a) $2\pi i$ (b) $-2\pi i$ (c) $4\pi i$ (d) 0.
Ans. (a)

14. The value of $\oint_{|z|=2} \left(\frac{e^z}{z} + \sin z\right) dz$ is equals to
(a) $2\pi ie$ (b) $-2\pi i$ (c) $4\pi i$ (d) 0
Ans. (d) use Cauchy Residue theorem

15. The value of $\int\limits_0^{2\pi} \frac{1}{13-5\sin\theta} d\theta$ is GATE(MA): 2004
(a)$-\frac{\pi}{6}$ (b)$-\frac{\pi}{12}$ (c)$\frac{\pi}{12}$ (d)$\frac{\pi}{6}$
Ans. (d) use the formula $\int\limits_0^{2\pi} \frac{1}{a+b\sin\theta} d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}, \ a > b > 0.$

16. The poles and residue at each pole of the function f(z)=cot$z$ is
(a) nπ, $n = \pm1, \pm2, \cdots$ *and* $Res = 1$ (b) nπ, n=0,$\pm1, \pm2, \cdots$ *and* $Res = 1$
(c) nπ, n=$\pm1, \pm2, \cdots$ *and*$Res = 2$ (d) nπ, n=$\pm1, \pm2, \cdots$ *and* $Res = \pm1$
Ans. (b) use the formula $Res(z = a) = \frac{\varphi(a)}{\psi(a)}$

17. The residue of $f(z) = \frac{ze^z}{(z-a)^3}$ at its pole is
(a) $e^a(1 + \frac{a}{2})$ (b)$e^a(1 - \frac{a}{2})$ (c)$e^a(1 + \frac{3a}{2})$ (d)$e^2(1 + \frac{a}{2})$.
Ans. (b) use the formula coefficient of $\frac{1}{t}$in $f(z)$ where$t = z - a$.

18. The integral $\oint_{|z|=2} \left(\frac{3z^2+11z-1}{z-4}\right) dz$ where C is the circle $|z| = 2$ travelled clockwise is
(a) $206\pi i$ (b)$2\pi i$ (c) $6\pi i$ (d) 0
Ans. (d) use Cauchy Theorem.

19. The integral $\oint_{|z|=2} \left(\frac{\cos z}{z^3}\right) dz$ equals to
(a) $2\pi ie$ (b)$-2\pi i$ (c) πi (d)$-\pi i$

Ans. (d)use Cauchy integral formulae

20. If $I = \oint_c (z-a)^n dz = 2\pi i$,[where C is the circle with center at a of radius R if
(a) $n \neq -1$, a inside C (b) $n \neq -1$, a outside C
(c) $n = -1$, a inside C (d) $n = -1$, a outside C
Ans. (c)

21. The value of the integral $\oint_{|z|=2} \frac{\cos(2\pi z)}{(92z-1)(z-3)} dz$ where C is the circle $|z| = 1$ is
(a) $-\pi i$ (b) $\frac{\pi i}{5}$ (c) $\frac{2\pi i}{5}$ (d) πi CE: 2009
Ans. (d)

22. The value of the integral $I = \oint_C \frac{\cos(\pi z)}{(z-i)^2} dz$ where C is the counter $4x^2 + y^2 = 2$. Then, I is equal to GATE(MA): 2003
(a) 0 (b)$-2\pi\, i$ (c) $2\pi\, i\left(\frac{\pi}{\sinh^2 \pi} - \frac{1}{\pi}\right)$ (d) $-\frac{2\pi^2 i}{\sinh^2 \pi}$
Ans. (d)

23. The contour C in the figure is described by $x^2 + y^2 = 16$. The value the integral $\oint \frac{z^2+8}{0.5z-1.5} dz$
(a) $-2\pi i$ (b) $2\pi i$ (c) $4\pi i$ (d) $-4\pi i$ GATE(MA): 2010
Ans. (d)

24. The value of the contour Integral $\oint_C \frac{dz}{z^2-2}$, $C : |z| = 4$ is equal to
(A) $\pi\, i$ (B) 0 (C)$-\pi\, i$ (D)$2\pi i$ GATE(MA): 2000
Ans. (B)

25. Given $f(z) = \frac{z}{(z-a)^2}$ with $|z| > a$, the residue of $f(z)\, z^{n-1}$ at $z = a$ for $n \geq 0$ will be
(A) a^{n-1} (B) a^n (C) na^n (D) na^{n-1}. EE: 2008
Ans. (D)

26. The value of $\oint_C \frac{dz}{(1+z^2)}$ where C is the contour $\left|z - \frac{i}{2}\right| = 1$ is
(A)$2\pi i$ (B) πi (C) $\tan^{-1} z$ (D) $\pi \tan^{-1} z$.
Ans. (B)

27. If $f(z) = c_0 + c_1 z^{-1}$, then $\int_{|z|=1} \frac{1+f(z)}{z} dz$ is given by
(A) $2\pi c_1$ (B) $2\pi (1 + c_0)$ (C) $2\pi i\, c_1$ (D) $2\pi i\ (1 + c_0)$.
Ans. (B).

28. The residue of the function $f(z) = \frac{1}{(z+2)^2 (z-2)^2}$ at $z = 2$ is
(A) $-\frac{1}{32}$ (B)$-\frac{1}{16}$ (C) $\frac{1}{16}$ (D) $\frac{1}{32}$.
Ans. (A) Since $Res(z = a) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]$.
So, $Res\,(z = 2) = \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left[(z-2)^2 \frac{1}{(z+2)^2 (z-2)^2}\right] = -1/32$.

29. For the function of a complex variable $W = \ln Z$, (where $W = u + iv$ and $Z = x + iy$) the u =constant lines get mapped in $Z-$plane as
(A) set of radial straight lines (B) set of concentric circles
(C) set of confocal hyperbolas (D)set of confocal ellipses.
Ans. (A)

30. Let D be the semi circular contour of radius 2, then the value of the integral $\oint_D \frac{1}{(s^2+1)} ds$ is ECE : 2007
(A) $i\pi$ (B) $-i\pi$ (C)$-\pi$ (D) π.

Ans. (A) Only the poles at $s = \pm i$ lies inside the contour. $Res(s = \pm i) = \pm\frac{1}{2}$. Therefore by Cauchy Residue theorem $\int_C f(z)dz = 2\pi i \sum_{r=1}^{n} Res = 0$.

31. An analytic function of a complex variable $z = x+iy$ is expressed as $f(z) = u(x, y)+iv(x, y)$ where $i = \sqrt{-1}$. If $u = xy$, the expression for v should be

(A) $\frac{(x+y)^2}{2} + k$ (B) $\frac{x^2-y^2}{2} + k$ (C) $\frac{y^2-x^2}{2} + k$ (D) $\frac{(x-y)^2}{2} + k$.
Ans. (C)

32. If $z = x + iy$, where x and y are real. The value of $\left|e^{iz}\right|$ is
(A) 1 (B) $e^{\sqrt{x^2+y^2}}$ (C) e^{y} (D) e^{-y}.
Ans. (D)

33. The value of $\int \frac{\sin z}{z} dz$, where the contour of integration is a simple closed curve around the origin, is
(A) 0 (B) $2\pi i$ (C) ∞ (D)$\frac{1}{2\pi i}$.
Ans. (A)

34. The analytic function $f(z) = \frac{z-1}{z^2+1}$ has singularities at
(A)1 and -1 (B) 1 and i (C) 1 and $- - i$ (D) i and $-i$.
Ans. (D)

35. Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral $\int_0^{\pi/2} \frac{\cos x+i\sin x}{\cos x-i\sin x} dx$?
(A)0 (B)2 (C)$-i$ (D) i. CS: 2011
Ans. (D) $\int_0^{\frac{\pi}{2}} \frac{e^{ix}}{e^{-ix}} dx = \int_0^{\frac{\pi}{2}} e^{2ix} dx = \left[\frac{e^{2ix}}{2i}\right]_0^{\frac{\pi}{2}} = \frac{1}{2i}\left(e^{i\pi} - 1\right) = \frac{1}{2i}(\cos\pi + i\sin\pi - 1\ ) = \frac{-2}{2i} = i$.

36. The value of $\oint_G \left(\frac{4}{z-1} - \frac{5}{z+4}\right) dz$, where G is the circle $|z| = 2$.
(a) $8\pi i$ (b) $-8\pi i$ (c) $4\pi i$ (d) 0.
Ans. (a) The point $z = -4$ lies outside $|z| = 2$, so the Cauchy–Goursat theorem shows that the second term in the integrand contributes nothing to the integral. Deforming G into any circle centered on $z = 1$ that does not contain the point $z = -4$.

37. If $f(z)$ is analytic in the entire z plane and bounded for all z, then $f(z)$ is
(a) constant (b) variable (c) not constant (d) any function of z.
Ans. (a) Liouville's theorem: If $f(z)$ is analytic in the entire z plane and bounded for all z, then $f(z)$ =constant.

38. If a function $f(z)$ is continuous in a domain D and such that of $\oint_G f(z)dz = 0$, for every simple contour G in D , then f(z) is
(a) constant (b) Analytic (c) not Analytic (d) any function of z.
Ans. (b) Morera's theorem: If a function $f(z)$ is continuous in a domain D and such that of $\oint_G f(z)dz = 0$, for every simple contour G in D, then $f(z)$ is analytic in D.

39. The product of two complex numbers $1 + i$ and $2 - 5i$ is ME: 2011
(A) $7 - 3i$ (B) $3 - 4i$ (C) $-3 - 4i$ (D) $7 + 3i$.
Ans. (A) $(1 + i)(2 - 5i) = 2 - 5i + 2i + 5 = 7 - 3i$.

40. If C is the positively oriented unit circle |z| = 1 and f(z) = $exp(2z)$, then $\oint_C \frac{f(z)}{z^4} dz$ is
(A) πi (B) $2\pi i$ (C) $\frac{8\pi i}{3}$ (D) $-4\pi i$.
Ans. (C)

41. The value of the integral of $\oint_C \bar{z}dz$, when C is the right-hand half $z = 2e^{i\theta}$ $\left(-\frac{\pi}{2} \le \theta \le -\frac{\pi}{2}\right)$ is
(A) πi (B) $2\pi i$ (C) $4\pi i$ (D) $-4\pi i$.
Ans. (C) Since $z = 2e^{i\theta}$ $\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$ of the circle |z| = 2, from z = -2/ to z = 2i. Therefore
$\bar{z} = 2e^{-i\theta}$ $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{2e^{i\theta}}\, d\left(2e^{i\theta}\right) = 4i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = 4\pi i$

42. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. The value of the integrals: $\oint_C \frac{\cos z}{z(z^2+8)} dz$ is
(A) $\frac{\pi i}{4}$ (B) $4\pi i$ (C) $-4\pi i$ (D) $-\frac{\pi i}{4}$.
Ans. (A)

43. The residue of $f(z) = \frac{z^3}{(z-2)(z-3)}$ at its poles at $z = 2$ and $z = 3$ respectively are
(A) 19, 12 (B) 1, 0 (C) −27, 8 (D) −8, 27.
Ans. (D). Since Res (z = a) = $\lim_{z\to a}(z-a)f(z)$, Therefore

$$\text{Res}\,(z=2) = \lim_{z\to 2}(z-2)\frac{z^3}{(z-2)(z-3)} = -8$$

$$\text{Res}\,(z=3) = \lim_{z\to 3}(z-3)\frac{z^3}{(z-2)(z-3)} = 27$$

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44. The residue of $f(z) = \frac{ze^z}{(z-a)^3}$ at its pole is
(A) $\frac{\pi i}{4}$ (B) $e^a\left(\frac{a}{2}+1\right)$ (C) $e^a\left(\frac{a}{2}-1\right)$ (D) $e^a(a+1)$. **Ans.** (B) Put $z = t$.
$f(z) = \frac{(t+a)^3}{a^3} = \left(\frac{a}{t^3} + \frac{1}{t^2}\right)e^{(a+t)} = e^a\left(\frac{a}{t^3}+\frac{1}{t^2}\right)\left(1+\frac{t}{1!}+\frac{t^2}{2!}+\cdots\right) = e^a\left(\frac{a}{2}+2\right)\frac{1}{t} + (a+1)\frac{1}{t^2} + \cdots$.
The Residue at $z = a$ is coefficient of $\frac{1}{t} = e^a\left(\frac{a}{2}+2\right)$

45. The value of the integral of $\oint_G \frac{4-3z}{z(z-1)(z-3)}dz$, where G is the circle $|z| = \frac{3}{2}$
(A) $\frac{\pi i}{4}$ (B) $2\pi i$ (C) $-4\pi i$ (D) $-\frac{\pi i}{4}$
Ans. (B).

46. The value of $\int_0^{\pi} \frac{1}{12-5\cos\theta}d\theta$ is
(a) $\frac{2\pi i}{5}$ (b) $\frac{2\pi}{5}$ (c) $\frac{4\pi}{13}$ (d) 0.
Ans. (c) use the formula $\int_0^{2\pi} \frac{1}{a+b\cos\theta}d\theta = \frac{2\pi}{\sqrt{a^2+b^2}}$.

47. The value of $\int_{|z|=3}\left(\frac{\cos z}{z}+\sin z\right)dz$ is
(A) $2\pi i\left(\frac{e^2}{2}+\sin 2\right)$ (B) $2\pi i\left(\frac{e^2}{2}+0\right)$ (C) $2\pi i$ (D) 0
Ans. (C)

48. The value of the contour integral $\frac{1}{2\pi i}\oint_C f(z)\,dz$ where $f(z) = \frac{z}{2}+\frac{1}{z}+\frac{2z}{z^2-1}$ and the contour C is the circle of radius 2 centered at the origin, traversed in the contour clockwise direction is

(A) 1 (B) $\frac{1}{2}$ (C) 1 (D) 3

Ans. (A) $\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\oint_C \left[\frac{z}{2}+\frac{1}{z}+\frac{2z}{z^2-1}\right]dz = \frac{1}{2\pi i}\oint_C \left[\frac{z}{2}\right]dz + \frac{1}{2\pi i}\oint_C \left[\frac{1}{z}+\frac{2z}{z^2-1}\right]dz$

$$= 0 + \text{Sum of Residue}$$

Now, $\frac{1}{2\pi i}\oint_C \left[\frac{z}{2}\right]dz = 0$, By Cauchy Theorem.
Since Res (z = a) = $\lim_{z\to a}(z-a)f(z)$. Therefore

$$\text{Res}\,(z=0) = \lim_{z\to 0}(z-0)\frac{1}{z} = 1\ ,$$

$$\text{Res}\,(z=1) = \lim_{z\to 1}(z-1)\frac{2z}{z^2-1} = 1$$

$$\text{Res}\,(z=-1) = \lim_{z\to 1}(z+1)\frac{2z}{z^2-1} = -1$$

49. Let f (z) $= \frac{\sin z}{z^2} - \frac{\cos z}{z}$ then
(A) f has a pole of order 2 at $z = 0$
(B) f has a simple pole at $z = 0$.
(C) $\oint_{|z|=1} f(z)\,dz = 0$, where the integral is taken anti-clockwise
(D) the residue of f at z = 0 is $-2\pi i$.
Ans. (B) Since $\lim_{z\to 0}\frac{\sin z}{z} = 1$. So, $\frac{\sin z}{z^2} \equiv \frac{1}{z}$ as $z \to 0$.

50. Let $P(z), Q(z)$ be two complex non-constant polynomials of degree m, n respectively. The number of roots of $P(z) = P(z)Q(z)$ counted with multiplicity is equal to
(a) Min $\{m, n\}$ (b) Max $\{m, n\}$ (c) $m+n$ NET(MS)(Jun): 2016 (d) $m-n$.
Ans. (c).

51. The Residue of the function $f(z) = e^{-e^{\frac{1}{z}}}$ at $z = 0$ is NET(MS)(Jun): 2016
(a) $1+e^{-1}$ (b) e^{-1} (c) $-e^{-1}$ (d) $1-e^{-1}$.
Ans. (c). Since $f(z) = e^{-e^{\frac{1}{z}}} = e^{-(1+\frac{1}{1!z}+\frac{1}{2!z^2}+\cdots)}$. So the coefficient of $\frac{1}{z}$ is $-1+\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots$.

52. Consider the function $F(z) = \int_1^2 \frac{1}{(x-z)^2}dx,\ Im(z) > 0$. Then there is a meromorphic function $G(z)$ on $\mathbb{C}$ that agree with $F(z)$ when $Im(z) > 0$, such that NET(MS)(Jun): 2016
(a) 1, ∞ are poles of $G(z)$ (b) 0, 1, ∞ are poles of $G(z)$
(c) 1, 2 are poles of $G(z)$ (d) 1, 2 are simple poles of $G(z)$.
Ans. (c) and (d).

53. Let f be a real valued harmonic function on $\mathbb{C}$, i.e., f satisfies the equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Defined the functions $g = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}$ and $h = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}$. Then NET(MS)(Jun): 2015
(a) g and h are both holomorphic functions.
(b) g is holomorphic but h need not be holomorphic.
(c) h is holomorphic but g need not be holomorphic.
(d) both h and g are identically equal to the zero functions.
Ans. (b). Let $g = u + iv$ where $u = \frac{\partial f}{\partial x}$ and $v = \frac{\partial f}{\partial y}$. Also $\frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$.

54. $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz =$ NET(MS)(Jun): 2015
(a) 0 (b) $-2\pi i$ (c) $2\pi i$ (d) 1.
Ans. (c). Since only $z = -2$ lies within the region $|z+1| = 2$. So, $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz = \int_{|z+1|=2} \Big(-1 + \frac{1}{2-z} + \frac{1}{2+z} \Big) dz = 0 + 0 + \int_{|z+1|=2} \frac{dz}{2+z} = 2\pi i.$

55. $\int_{|z-3i|=2} \frac{dz}{z^2+4} =$ GATE(MA): 2008
(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $-\frac{i\pi}{2}$ (d) $\frac{i\pi}{2}$.
Ans. (b). Since only $z = 2i$ lies within the region $|z-3i| = 2$. So, $\int_{|z-3i|=2} \frac{dz}{z^2+4} = 2\pi i \times \lim_{z\to 2i} (z-2i) \frac{1}{z^2+4} = \frac{\pi}{2}$.

56. Let f be an entire function. Which of the following statements are correct.
(a) f is constant if the range of f is contained in a straight line. NET(MS)(Jun): 2015
(b) f is constant if f has uncountable many zeros.
(c) f is constant if f is bounded on $\{z \in \mathbb{C} : Re(z) \le 0\}$
(d) f is constant if the real part of f is constant.
Ans. (a), (b) and (d).

57. Let p be a polynomial in 1− complex variable. Suppose all zeroes of p are in the upper half plane $H = \{z \in \mathbb{C} \big| Im(z) \ge 0\}$. Then NET(MS)(Jun): 2015
(a) $Im \frac{p'(z)}{p(z)} > 0$ for $z \in \mathbb{R}$ (b) $Re\, i \frac{p'(z)}{p(z)} < 0$ for $z \in \mathbb{R}$.
(c) $Im \frac{p'(z)}{p(z)} > 0$ for $z \in \mathbb{C}$ with $Im(z) < 0$. (d) $Im \frac{p'(z)}{p(z)} > 0$ for $z \in \mathbb{C}$ with $Im(z) > 0$.
Ans. (a), (b) and (c).

58. Consider the following power series series in the complex variables z: $f(z) = \sum_{n=1}^{\infty} n \log n z^n$, $g(z) = \sum_{n=1}^{\infty} \frac{e^{n^2}}{n} z^n$. If r and R are the radii of convergence of f and g respectively, then
(a) $r = 0, R = 1$ (b) $r = 1, R = 0$ (c) $r = 1, R = \infty$ (d) $r = \infty, R = 1$.
Ans. (b). NET(MS)(Dec.): 2015

59. The bilinear transformation w which maps the points $0, 1, \infty$ in the $z-$plane onto the points $-i, -\infty, 1$ in the $w-$plane is GATE(MA): 2003
(a) $\frac{z-1}{z+i}$ (b) $\frac{z-i}{z+1}$ (c) $\frac{z+i}{z-1}$ (d) $\frac{z+1}{z-i}$
Ans. (d). Since the bilinear transformation w which maps the points $0, 1, \infty$ in the $z-$plane onto the points $-i, -\infty, 1$ in the $w-$plane is give by
$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \Rightarrow \frac{(w-0)(1-\infty)}{(0-1)(\infty-w)} = \frac{(z+i)(\infty-1)}{(-i-\infty)(1-z)}$.

60. The bilinear transformation w which maps the points $-1, 0, 1$ in the $z-$plane onto the points $-i, 1, i$ in the $w-$plane. Then $f(1-i)$ equals GATE(MA): 2004
(a) $-1+2i$ (b) $2i$ (c) $-2+i$ (d) $-1+i$
Ans. (c).

61. The number of zeros, counting multiplicities of the polynomial $z^5 + 3z^3 + z^2 + 1$ inside the circle $|z| = 2$ is GATE(MA): 2004
(a) 0 (b) 2 (c) 3 (d) 5.
Ans. (d). Let $F(z) = z^5 + 3z^3 + z^2 + 1$ be the complex polynomial and the circle $|z| = 2$, then zero's inside the circle are defined by $F(z) = f(z) + g(z)$, where $g(z) = z^5$ and $f(z) = 3z^3 + z^2 + 1$. Then $\left| \frac{f(z)}{g(z)} \right| \le \frac{3 \cdot 2^3 + 2^2 + 1}{2^5} = \frac{29}{32} < 1$. Therefore $|f(z)| < |g(z)| \Rightarrow F(z)$ has all five zero's in $|z| = 2$.

62. The number of roots of the equation $z^5 - 12z^2 + 14 = 0$ that lie in the region $\{z \in \mathbb{C} : 2 \leq |z| < \frac{5}{2}\}$ is GATE(MA): 2005
(a) 2 (b) 3 (c) 4 (d) 5.
Ans. (d). Let $g(z) = z^5$ and $f(z) = -12z^2 + 14$. Then $\left|\frac{f(z)}{g(z)}\right| < 1$. Therefore the number of the roots of the equation is 5.

63. The bilinear transformation w which maps the points $-1, i, -i$ in the $z-$plane onto the points $1, \infty, 0$ in the $w-$plane. Then $f(1)$ is equal to GATE(MA): 2008
(a) -2 (b) -1 (c) i (d) $-i$
Ans. (b). Since the bilinear transformation $w = f(z)$ is give by
$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$.

64. Let $a, b, c, d \in \mathbb{R}$ be such that $ad - bc > 0$. consider the Mobius Transformation $T_{a,b,c,d}(z) = \frac{az+b}{cz+d}$. Define NET(MS)(Dec.): 2015
$H_+ = \{z \in \mathbb{C} : Im(z) > 0\}$, $H_- = \{z \in \mathbb{C} : Im(z) < 0\}$.
$R_+ = \{z \in \mathbb{C} : Re(z) > 0\}$, $R_- = \{z \in \mathbb{C} : Re(z) < 0\}$.
Then, $T_{a,b,c,d}$ maps
(a) H_+ to H_+ (b) H_+ to H_- (c) R_+ to R_+ (d) R_+ to R_-.
Ans. (a).

65. Let $w(z) = \frac{az+b}{cz+d}$ and $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ be bilinear (Mobius) transformations. Then, the following is also a bilinear transformation GATE(MA): 2002
(a) $f(z)w(z)$ (b) $f(w(z))$ (c) $f(z) + g(z)$ (d) $f(z) + \frac{1}{w(z)}$
Ans. (b).

66. Let $f(z) = \frac{1}{e^z-1}$ for all $z \in \mathbb{C}$ such that $e^z \neq 1$. Then NET(MS)(Dec.): 2015
(a) f is meromorphic (b) the only singularities are poles
(c) f has infinitely many poles in the imaginary axis (d) each pole of f is simple
Ans. (a), (b), (c) and (d).

67. Let f be a analytic function in $\mathbb{C}$. Then f is constant if the zero set of f contains the sequence NET(MS)(Dec.): 2015
(a) $a_n = \frac{1}{n}$ (b) $a_n = (-1)^{n-1}\frac{1}{n}$ (c) $a_n = \frac{1}{2n}$ (d) $a_n = n$ if 4 does not divide n and $a_n = \frac{1}{n}$ if 4 divides n.
Ans. (a), (b), (c) and (d).

68. Consider the function $f(z) = \frac{1}{z}$ on the annulus $A = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$. Which of the following is / are true? NET(MS)(Dec.): 2015
(a) There is a sequence $p_n(z)$ of polynomials that approximate $f(z)$ uniformly on compact subsets of A.
(b) there is a sequence $r_n(z)$ of rational functions whose poles are contained in $\mathbb{C}\backslash\mathbb{A}$ and which approximate $f(z)$ uniformly on compact subsets of A.
(c) No sequence $p_n(z)$ of polynomials approximate $f(z)$ uniformly on compact subsets of A.
(d) No sequence $r_n(z)$ of rational functions whose poles are contained in $\mathbb{C}\backslash\mathbb{A}$, approximate $f(z)$ uniformly on compact subsets of A.
Ans. (b) and (c).

69. The straight lines $L_1 : x = 0$, $L_2 : y = 0$ and $L_3 : x + y = 1$ are mapped by transformation

$w = z^2$ into the curves C_1, C_2 and C_3 respectively. The angle of intersection between the curves at $w = 0$ is GATE(MA): 2012
(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π
Ans. (c). Since $w = z^2 = (x + iy)^2$. $C_1 : w = -y^2$, $C_2 : w = x^2$ and $C_3 : w = (x + i(1 - x))^2 = -1 + 2x + 2ix(1 - x)$. So angle between curves (are C_1 and C_2) at $w = 0$ is $\frac{\pi}{2}$.

70. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic except for a simple pole at $z = 0$ and let $g : \mathbb{C} \to \mathbb{C}$ be analytic. Then, the value of $\frac{\lim\limits_{z\to 0} Res\ f(z)g(z)}{\lim\limits_{z\to 0} Res\, f(z)}$ is GATE(MA): 2011
(a) $g(0)$ (b) $g'(0)$ (c) $\lim\limits_{z\to 0} zf(z)$ (d) $\lim\limits_{z\to 0} zf(z)g(z)$
Ans. (a). Since $\frac{\lim\limits_{z\to 0} Res\ f(z)g(z)}{\lim\limits_{z\to 0} Res\, f(z)} = \frac{\lim\limits_{z\to 0} zf(z)g(z)}{\lim\limits_{z\to 0} zf(z)} = \lim\limits_{z\to 0} g(z)\frac{\lim\limits_{z\to 0} zf(z)}{\lim\limits_{z\to 0} zf(z)} = g(0)$.

71. Let $u(x, y) = 2x(1 - y)$ for all real x and y. Then a function $v(x, y)$, so that $f(z) = u(x, y) + iv(x, y)$ is analytic is GATE(MA): 2010
(a) $x^2 - (y - 1)^2$ (b) $(x - 1)^2 - y^2$ (c) $(x - 1)^2 + y^2$ (d) $x^2 + (y - 1)^2$
Ans. (a).

72. Let $f(z)$ be analytic on $D = \{z \in \mathbb{C} : |z - 1| < 1\}$ such that $f(1) = 1$. If $f(z) = f(z^2)$, $\forall z \in D$, then which one of the following statements is not correct? GATE(MA): 2010
(a) $f(z) = [f(z)]^2$, $\forall z \in D$ (b) $f(\frac{z}{2}) = \frac{f(z)}{2}$, $\forall z \in D$
(c) $f(z^3) = [f(z)]^3$, $\forall z \in D$ (d) $f'(1) = 0$.
Ans. (a). Since $f(z) = f(z^2)$, $\forall z \in D$, so $f(z) \neq [f(z)]^2$, $\forall z \in D$.

73. For the function $f(z) = \sin\left(\frac{1}{\cos(\frac{1}{z})}\right)$, the point $z = 0$ is GATE(MA): 2009
(a) a removable singularity (b) a pole
(c) an essential singularity (d) a non-isolated singularity
Ans. (c). Since $f(z) = \sin\left(\frac{1}{\cos(\frac{1}{z})}\right) = 0 \Rightarrow \frac{1}{\cos(\frac{1}{z})} = n\pi,\ n \in \mathbb{Z}$.
So, $\cos(\frac{1}{z}) \to 0$ as $n \to \infty \Rightarrow z = \frac{2}{(2n+1)\pi}$, $n \in \mathbb{Z}$.
So, $\mathbb{Z} \to 0$ as $n \to \infty$. Hence $z = 0$ is an essential singularity.
Note: It is also called isolated essential singularity.

74. For the function $f(z) = \cot\left(\frac{1}{\cos(\frac{1}{z})}\right)$, the point $z = 0$ is
(a) a removable singularity (b) a pole
(c) an isolated essential singularity (d) a non-isolated essential singularity
Ans. (d). Here $f(z) = \frac{\cos\left(\frac{1}{\cos(\frac{1}{z})}\right)}{\sin\left(\frac{1}{\cos(\frac{1}{z})}\right)}$. So, $\frac{1}{f(z)} = 0 \Rightarrow \sin\left(\frac{1}{\cos(\frac{1}{z})}\right) = 0 \Rightarrow \frac{1}{\cos(\frac{1}{z})} = n\pi,\ n \in \mathbb{Z}$.
So, $\cos(\frac{1}{z}) \to 0$ as $n \to \infty \Rightarrow z = \frac{2}{(2n+1)\pi}$, $n \in \mathbb{Z}$.
So, $\mathbb{Z} \to 0$ as $n \to \infty$. Hence $z = 0$ is a non- isolated essential singularity.
Note: It is note that the numerator of $f(z)$ is zero implies the isolated essential singularity, but the denominator of $f(z)$ is zero implies the non-isolated essential singularity.

75. For the function $f(z) = \tan\left(\frac{1}{\cos(\frac{1}{z})}\right)$, the point $z = 0$ is
(a) a removable singularity (b) a pole
(c) an isolated essential singularity (d) a non-isolated essential singularity
Ans. (c).

76. Let $f(z) = \sum_{n=0}^{15} z^n$ for $z \in \mathbb{C}$. If $\mathbb{C} : |z-i| = 2$, then $\oint_{\mathbb{C}} \frac{f(z)dz}{(z-i)^{15}}$ is equal to GATE(MA): 2009
(a) $2\pi i(1+15i)$ (b) $2\pi i(1-15i)$ (c)$4\pi i(1+15i)$ (d) $2\pi i$
Ans. (a). Here $\frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{f(z)dz}{(z-i)^{15}} = \lim_{z\to i} Res f(z) = \frac{f^{14}(i)}{14!} = \frac{14!+15!i}{14!} = 1+15i$.

77. For the function $f(z) = \sin\frac{1}{z}$, $z=0$ is a GATE(MA): 2002
(a) a removable singularity (b) simple pole
(c) branch point (d) an essential singularity
Ans. (d).

78. For example of a function with a non-isolated essential singularity at $z=2$ is GATE(MA): 2003
(a) $\tan\frac{1}{z-2}$ (b) $\sin\frac{1}{z-2}$ (c) $e^{(z-2)}$ (d) $\tan\frac{1z-2}{z}$
Ans. (a). Since $\cos\frac{1}{z-2} = 0$ gives us the non-isolated essential singularity.

79. Let S be the open unit disk and $f : S \to \mathbb{C}$ be a real valued analytic function with $f(0)=1$. Then, the set $\{z \in S : f(z) \neq 1\}$ is GATE(MA): 2008
(a) empty (b) non-empty finite (c) countably infinite (d) uncountable
Ans. (a).

80. Let $S = \{0\} \bigcup \left\{\frac{1}{4n+7} : n = 1, 2, \cdots\right\}$. Then, the number of analytic functions which vanish only on S is GATE(MA): 2007
(a) infinite (b) 0 (c) 1 (d) 2
Ans. (b). Since $\bar{S} = S$, so S is closed. If possible let $f(z)$ be analytic in $\bar{S}$. But limit point of zero's is an isolated essential singularity, so '0' can not be zero of $f(z)$. Hence, there is no such analytic function which vanish only on S. So number of analytic function is 0.

81. It is given that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 3+4i$. Then, the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is GATE(MA): 2007
(a) ≤ 5 (b) ≥ 5 (c) < 5 (d) > 5.
Ans. (b). Since $|z-0| \leq R \Rightarrow |3+4i-0| \leq R \Rightarrow R \geq 5$.

82. The principal value of $\log\left(i^{\frac{1}{4}}\right)$ is GATE(MA): 2005
(a) πi (b) $\frac{\pi i}{2}$ (c) $\frac{\pi i}{4}$ (d) $\frac{\pi i}{8}$
Ans. (d). Since $z = \frac{1}{4}\log i = \frac{1}{4}\log e^{\frac{i\pi}{2}} = \frac{\pi i}{8}$.

83. Consider the functions $f(z) = x^2 + iy^2$ and $g(z) = x^2 + y^2 + ixy$. At $z=0$, GATE(MA): 2005
(a) f is analytic but not g (b) g is analytic but not f
(c) both f and g are analytic (d) neither f nor g is analytic
Ans. (d).

84. The coefficient of $\frac{1}{z}$ in the expansion of $\log\left(\frac{z}{z+1}\right)$, valid in $|z| > 1$ is GATE(MA): 2005
(a) -1 (b) 1 (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$
Ans. (a). Since $\log\left(\frac{z}{z+1}\right) = -\log(1+\frac{1}{z}) = -\left(\frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \cdots\right)$.

85. If D is the open unit disk in $\mathbb{C}$ and $f : \mathbb{C} \to D$ is analytic with $f(10) = \frac{1}{2}$, then $f(10+i)$ is
(a) $\frac{1+i}{2}$ (b) $\frac{1-i}{2}$ (c) $\frac{1}{2}$ (d) $\frac{i}{2}$ GATE(MA): 2004
Ans. (c). Since every entire and bounded function is constant(By Liouville's theorem).

86. The real part of the principal value of 4^{4-i} is GATE(MA): 2004
(a) $256\cos(\ln 4)$ (b) $64\cos(\ln 4)$ (c) $16\cos(\ln 4)$ (d) $4\cos(\ln 4)$
Ans. (a). Since $4^{4-i} = e^{4-i}\log 4 = e^{4\log 4}\cdot e^{-i\log 4} = 4^4\Big(\cos(\ln 4) + i\sin(\ln 4)\Big)$.

87. Consider a function $f(z) = u + iv$ defined on $|z-1| < 1$ where u, v are real-valued functions of x, y. Then, $f(z)$ is analytic for u equals to GATE(MA): 2003
(a) $x^2 + y^2$ (b) $\ln(x^2 + y^2)$ (c) e^{xy} (d) $e^{x^2-y^2}$
Ans. (b) Since $u = \ln(x^2 + y^2)$ has been satisfied by the equation $\nabla^2 u = 0$.

88. At $z = 0$, the function $f(z) = z^2\bar{z}$ GATE(MA): 2003
(a) does not satisfy Cauchy-Riemann equations
(b) satisfies Cauchy-riemann equations but is not differentiable
(c) is differentiable
(d) is analytic
Ans. (a)

89. The function $f(z) = z^2$ maps the first quadrant onto GATE(MA): 2002
(a) itself (b) upper half plane (c) third quadrant (d) right half plane
Ans. (b). Here $U = x^2 - y^2$ and $V = 2xy$. Since in the first quadrant we have $x \geq 0$, $y \geq 0$. So $v \geq 0$ but $u \leq 0$ or ≥ 0.

90. The radius of convergence of the power series of the function $f(z) = \frac{1}{1-z}$ about $z = \frac{1}{z}$ is
(a) 1 (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) 0. GATE(MA): 2002
Ans. (c). Here $f(z) = \frac{1}{1-z} = \frac{1}{1-\frac{1}{4}-(z-\frac{1}{4})} = \frac{4}{3}\Big(1 - \frac{4}{3}(z-\frac{1}{4})\Big)^{-1} = \frac{4}{3}\Big(1 + \frac{4}{3}(z-\frac{1}{4}) + \frac{4^2}{3^2}(z-\frac{1}{4})^2 + \cdots\Big)$. So $R = \frac{3}{4}$.

91. Let T be any circle enclosing the origin and oriented counter-clockwise. Then the value of the integral $\oint_T \frac{\cos z}{z^2}dz$ is GATE(MA): 2002
(a) $2\pi i$ (b) 0 (c) $-2\pi i$ (d) undefined
Ans. (b). Since $\oint_T \frac{\cos z}{z^2}dz = 2\pi i f'(0) = -2\pi i \sin z\Big|_{z=0} = 0$.

92. The function $\sin z$ is analytic in GATE(MA): 2001
(a) $\mathbb{C}\bigcup\{\infty\}$ (b) $\mathbb{C}$ expect on the negative real axis
(c) $\mathbb{C}\bigcap\{\infty\}$ (d) $\mathbb{C}$
Ans. (d)

93. If $f(z) = z^3$, then it GATE(MA): 2001
(a) has an essential singularity at $z = \infty$ (b) has a pole of order 3 at $z = \infty$
(c) has a pole of order 3 at $z = 0$ (d) is analytic at $z = \infty$.
Ans. (b).

94. Let $\int_C \Big[\frac{1}{(z-2)^4} - \frac{(a-2)^2}{z} + 4\Big]dz = 4\pi$, where the close curve $\mathbb{C}$ is the triangle having vertices at $i, \frac{-1-i}{\sqrt{2}}$ and $\frac{1-i}{\sqrt{2}}$. The integral being taken in anti-clockwise direction. Then, one value of a is GATE(MA): 2012
(a) $1+i$ (b) $2+i$ (c) $3+i$ (d) $4+i$.
Ans. (c). Now, by Cauchy's integral formula, $\int_C \frac{1}{(z-2)^4}dz = 0$, $\int_C \frac{(a-2)^2}{z}dz = 2\pi i(a-2)^2$ and $\int_C 4dz = 0$. Hence we get, $0 - 2\pi i(a-2)^2 + 0 = 4\pi$. Therefore $a = 3+i$.

95. Consider the functions $f(z) = \frac{z^2+\alpha z}{(z+1)^2}$ and $g(z) = \sinh(z - \frac{\pi}{2\alpha})$, $\alpha \neq 0$. The residue of $f(z)$ at

its pole is equal to 1. Then the value of α is GATE(MA): 2012
(a) -1 (b) 1 (c) 2 (d) 3.
Ans. (d).

96. Consider the functions $f(z) = \frac{z^2+\alpha z}{(z+1)^2}$ and $g(z) = \sinh(z - \frac{\pi}{2\alpha})$, $\alpha \neq 0$. For the value of α the function $g(z)$ is not conformal at a point GATE(MA): 2012
(a) $\frac{\pi(1+3i)}{6}$ (b) $\frac{\pi(3+i)}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi i}{2}$.
Ans. (a). Since $g(z)$ is not conformal, if $g'(z) = 0 \Rightarrow \cosh(z - \frac{\pi}{6}) = 0$.

97. Let $f(z)$ be an entire function that $|f(z) \leq K|z|$, $\forall z \in \mathbb{C}$, for some $K > 0$. If $f(1) = i$, the value of $f(i)$ is GATE(MA): 2011
(a) 1 (b) -1 (c) i (d) $-i$.
Ans. (b). Let $f(z) = kz$.

98. For the function $f(z) = \frac{z}{8-z^3}$, $z = x + iy$, $\lim\limits_{z\to 2} Res f(z)$ is GATE(MA): 2011
(a) $-\frac{1}{8}$ (b) $\frac{1}{8}$ (c) $-\frac{1}{6}$ (d) $\frac{1}{6}$.
Ans. (c).

99. The Cauchy principal value of $\int\limits_{-\infty}^{\infty} \frac{x}{8-x^3}dx$ is GATE(MA): 2011
(a) $-\frac{\sqrt{3}\pi}{6}$ (b) $-\frac{\sqrt{3}\pi}{8}$ (c) $\pi\sqrt{3}$ (d) $-\pi\sqrt{3}$.
Ans. (a). Let, $\int\limits_{-\infty}^{\infty} \frac{z}{8-z^3}dz$. Therefore the poles are $z = 2, -1 \pm \sqrt{3}i$. Find the Res. and use the formula.

100. Let $\oint_C \frac{f(z)}{(z-1)(z-2)}$ where $f(z) = \sin\frac{\pi z}{2} + \cos\frac{\pi z}{2}$ and C is the curve $|z| = 3$ oriented anti-clockwise. Then the value of I is GATE(MA): 2010
(a) $4\pi i$ (b) 0 (c) $-2\pi i$ (d) $-4\pi i$
Ans. (d).

101. Let $\sum\limits_{n=-\infty}^{\infty} b_n z^n$ be the Laurent series expansion of the function $\frac{1}{z\sinh z}$, $0 < |z| < \pi$. Then which one of the following is correct? GATE(MA): 2010
(a) $b_{-2} = 1$, $b_0 = -\frac{1}{6}$, $b_2 = \frac{7}{360}$ (b) $b_{-3} = 1$, $b_{-1} = -\frac{1}{6}$, $b_1 = \frac{7}{360}$
(c) $b_{-2} = 0$, $b_0 = -\frac{1}{6}$, $b_2 = \frac{7}{360}$ (d) $b_0 = 1$, $b_2 = -\frac{1}{6}$, $b_1 = \frac{7}{360}$
Ans. (a). Let $\sum\limits_{n=-\infty}^{\infty} b_n z^n = \frac{1}{z\sinh z} = \frac{2}{z(e^z - e^{-z})} = \frac{2}{2\left[(1+z+\frac{z^2}{2!}+\cdots)-(1-z+\frac{z^2}{2!}-\cdots)\right]}$.

102. Under the transformation $w = \sqrt{\frac{1-iz}{z-i}}$, the region $D = \{z \in \mathbb{C} : |z| < 1\}$ is transformed to
(a) $\{z \in \mathbb{C} : 0 < arg(z) < \pi\}$ GATE(MA): 2010
(b) $\{z \in \mathbb{C} : -\pi < arg(z) < 0\}$
(c) $\{z \in \mathbb{C} : 0 < arg(z) < \frac{\pi}{2}$ or $0 < arg(z) < \frac{3\pi}{2}\}$
(d) $\{z \in \mathbb{C} : \frac{\pi}{2} < arg(z) < \pi$ or $\frac{3\pi}{2} < arg(z) < 2\pi\}$
Ans. (d).

103. Let $\sum\limits_{-\infty}^{\infty} a_n(z+1)^n$ be the Laurent series expansion of $f(z) = \sin(\frac{z}{z+1})$. Then a_{-2} is equal to
(a) 1 (b) 0 (c) $\cos(1)$ (d) $-\frac{1}{2}\sin(1)$. GATE(MA): 2009
Ans. (b).

104. Let $u(x, y)$ be the real part of an entire function $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy \in \mathbb{C}$. If $\mathbb{C}$ is the positive oriented boundary of a rectangular region R in $\mathbb{R}^2$, then $\oint_{C} \left[u_y dx - u_x dy\right]$ is equal to GATE(MA): 2009
(a) 1 (b) 0 (c) 2π (d) π.
Ans. (b).

105. For the function $f(z) = \frac{e^{iz}}{z(z^2+1)}$, the residue of f at the isolated singular point in the upper half plane $\{z = x + iy \in \mathbb{C},\ y > 0\}$ $\lim_{z\to 2} Res f(z)$ is GATE(MA): 2009
(a) $-\frac{1}{2e}$ (b) $-\frac{1}{e}$ (c) $\frac{e}{2}$ (d) 1.
Ans. (a).

106. The Cauchy principal value of $\int\limits_{-\infty}^{\infty} \frac{\sin x dx}{x(x^2+1)}$ is GATE(MA): 2009
(a) $-2\pi(1 + 2e^{-1})$ (b) $\pi(1 - e^{-1})$ (c) $2\pi(1 + e)$ (d) $-\pi(1 + e^{-1})$.
Ans. (b). Since $\int\limits_{-\infty}^{\infty} \frac{\sin x dx}{x(x^2+a^2)} = \frac{\pi}{a^2}(1 - e^{-1})$.

107. Let $f(z) = \cos z - \frac{\sin z}{z}$ for non-zero $z \in \mathbb{C}$ and $f(0) = 0$. Then $f(z)$ has a zero at $z = 0$ of order
(a) 0 (b) 1 (c) 2 (d) greater than 2. GATE(MA): 2008
Ans. (c). Let us consider order m. Then find minimum value of m, for which $\lim_{z\to 0} \frac{f(z)}{z^m}$ exist.

108. Let $f(z) = \cos z - \frac{\sin z}{z}$ for non-zero $z \in \mathbb{C}$ and $f(0) = 0$ and let $g(z) = \sinh z$ for $z \in \mathbb{C}$. Then $\frac{g(z)}{z f(z)}$ has a pole at $z = 0$ of order GATE(MA): 2008
(a) 1 (b) 2 (c) 3 (d) greater than 3.
Ans. (b).

109. The fixed points of $f(z) = \frac{2iz+5}{z-2i}$ are
(a) $1 \pm i$ (b) $1 \pm 2i$ (c) $2i \pm 1$ (d) $i \pm 1$ GATE(MA): 2001
Ans. (c). For fixed points, we have $f(z) = z$.

110. For the function $f(z) = \frac{1-e^{-z}}{z}$, the point $z = 0$ is GATE(MA): 2000
(a) an essential singularity (b) a pole of order zero
(c) a pole of order one (d) a removal singularity
Ans. (b) and (d).

111. The transformation $w = e^{i\theta}\left(\frac{z-\rho}{\bar{\rho}z-1}\right)$, where ρ is a constant, maps $|z| < 1$ onto GATE(MA): 2000
(a) $|w| < 1,\ |\rho| < 1$ (b) $|w| > 1,\ |\rho| > 1$ (c) $|w| = 1,\ |\rho| = 1$ (d) $|w| = 3,\ \rho = 0$
Ans. (a).

112. Let $f(z)$ be an analytic function with a simple pole at $z = 1$ and a double pole at $z = 2$ with residues 1 and -2 respectively. Further, if $f(0) = 0$, $f(3) = -\frac{3}{4}$ and f is bounded as $z \to \infty$, then $f(z)$ must be GATE(MA): 2003
(a) $z(z-3) - \frac{1}{4} + \frac{1}{z-1} - \frac{2}{(z-1)^2} + \frac{1}{(z-2)^2}$ (b) $-\frac{1}{4} + \frac{1}{z-1} - \frac{2}{(z-1)^2} + \frac{1}{(z-2)^2}$
(c) $\frac{1}{z-1} - \frac{2}{(z-1)^2} + \frac{5}{(z-2)^2}$ (d) $\frac{15}{4} + \frac{1}{z-1} + \frac{2}{z-2} - \frac{7}{(z-2)^2}$
Ans. (d). According to the problem $\lim_{z\to 1}(z-1)f(z) = 1$, $\lim_{z\to 2} \frac{d}{dx}\{(z-2)^2 f(z)\} = -2$ and $\lim_{z\to\infty} f(z)$ is bounded.

113. Let $f(z) = u(x,y) + iv(x,y)$ be an entire function having Taylor's series expansion as $\sum_{n=0}^{\infty} a_n z^n$. If $f(x) = u(x,0)$ and $f(iy) = iv(0,y)$, then GATE(MA): 2003
(a) $a_{2n} = 0, \forall n$ (b) $a_0 = a_1 = a_2 = a_3 = 0, a_4 \neq 0$ (c) $a_{2n+1} = 0, \forall n$ (d) $a_0 \neq 0$ but $a_2 = 0$
Ans. (a).

114. In the Laurent series expansion of $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$ valid in the region $|z| > 2$, the coefficient of $\frac{1}{z^2}$ is GATE(MA): 2004
(a) -1 (b) 0 (c) 1 (d) 2
Ans. (a). Since $|z| > 2$ so, $|\frac{1}{z}| < \frac{1}{2} < 1$ and $|\frac{2}{z}| < 1$.
Therefore $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\Big[(1-\frac{1}{z})^{-1} - (1-\frac{2}{z})^{-1}\Big] = -\frac{1}{z^2} - \frac{3}{z^3} - \cdots$.

115. The principal value of the improper integral $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$ is GATE(MA): 2003
(a) $\frac{\pi}{e}$ (b) πe (c) $\pi + e$ (d) $\pi - e$
Ans. (a). Since $\int_{-\infty}^{\infty} \frac{\cos mx}{a^2+x^2} dx = \frac{\pi}{a} e^{-ma}$.

116. the value of $\int_{0}^{2\pi} \exp(e^{i\theta} - i\theta) d\theta$ equals to GATE(MA): 2006
(a) $2\pi i$ (b) 2π (c) π (d) πi
Ans. (b).

117. Which of the following is not the real part of the analytic function? GATE(MA): 2006
(a) $x^2 - y^2$ (b) $\frac{1}{1+x^2+y^2}$ (c) $\cos x cosh y$ (d) $x + \frac{x}{x^2+y^2}$
Ans. (b). Since $\nabla^2(\frac{1}{1+x^2+y^2}) \neq 0$.

118. The radius of convergence of $\sum_{n=0}^{\infty} \frac{(1+\frac{1}{n})^{n^2}}{n^3} z^n$ is GATE(MA): 2006
(a) e (b) $\frac{1}{e}$ (c) 1 (d) ∞.
Ans. (b). Since $\frac{1}{R} = \lim_{z\to\infty} \Big(\frac{(1+\frac{1}{n})^{n^2}}{n^3}\Big)^{\frac{1}{n}} = \lim_{z\to\infty} \frac{(1+\frac{1}{n})^n}{(n^{\frac{1}{n}})^3} = \frac{e}{1}$

119. The sum of the residue at all the poles of $f(z) = \frac{\cot \pi z}{(z+a)^2}$, where a is a constant, ($a \neq 0, \pm 1, \pm 2, \cdots$) is GATE(MA): 2006
(a) $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi\, cosec^2 \pi a$ (b) $-\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi\, cosec^2 \pi a$
(c) $-\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi\, cosec^2 \pi a$ (d) $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi\, cosec^2 \pi a$
Ans. (a). Since $f(z) = \frac{\cot \pi z}{(z+a)^2}$ has a poles at $z = -a$ of order 2 and at $z = n$, $n \in \mathbb{Z}$. So Res of $f(z)$ (at $z = -a$)$= -\pi cosec^2 \pi a$ and Res of $f(z)$ (at $z = n$) $= \lim_{z\to n} (z-n) \frac{\cos \pi z}{\sin \pi z (z+a)^2} = \frac{i}{\pi (n+a)^2}$.
Hence the sum of the residue at all the poles of $f(z) = \frac{\cot \pi z}{(z+a)^2}$ is $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi\, cosec^2 \pi a$.

120. Let $\mathbb{C}$ be the boundary of the triangle formed by the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Then, the value of the line integral $\oint_{\mathbb{C}} -2y dx + (3x - 4y^2) dy + (z^2 + 3y) dz$ is GATE(MA): 2007
(a) 0 (b) 1 (c) 2 (d) 4
Ans. (a).

121. Let $f(z) = 2z^2 - 1$. Then the maximum value of $|f(z)|$ on the unit disc $D = \{z \in C : |z| =\leq 1\}$ equals to GATE(MA): 2007
(a) 1 (b) 2 (c) 3 (d) 4
Ans. (c).

122. Let $f(z)$ be an analytical function. Then the value of $\int_0^{2\pi} f(e^{it}) \cos t dt$ equals to
(a) 0 (b) $2\pi f(0)$ (c) $2\pi f'(0)$ (d) $\pi f'(0)$ GATE(MA): 2007
Ans. (c).

123. Let G_1 and G_2 be the images of the disc $\{z \in \mathbb{C} | z + 1| < 1\}$ under the transformations $\omega = \frac{(1-i)z+2}{(1+i)z+2}$ and $\omega = \frac{(1+i)z+2}{(1-i)z+2}$ respectively. Then, GATE(MA): 2007
(a) $G_1 = \{\omega \in \mathbb{C} : Im(\omega) < 0\}$ and $G_2 = \{\omega \in \mathbb{C} : Im(\omega) > 0\}$
(b) $G_1 = \{\omega \in \mathbb{C} : Im(\omega) > 0\}$ and $G_2 = \{\omega \in \mathbb{C} : Im(\omega) < 0\}$
(c) $G_1 = \{\omega \in \mathbb{C} : Im(\omega) > 2\}$ and $G_2 = \{\omega \in \mathbb{C} : Im(\omega) < 2\}$
(d) $G_1 = \{\omega \in \mathbb{C} : Im(\omega) < 2\}$ and $G_2 = \{\omega \in \mathbb{C} : Im(\omega) > 2\}$
Ans. (b).

124. Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary analytic function satisfying $f(0) = 0$ and $f(1) = 2$. Then,
(a) there exist a sequence $\{Z_n\}$ such that $|Z_n| > n$ and $|f(Z_n)| < n$ GATE(MA): 2007
(b) there exist a sequence $\{Z_n\}$ such that $|f(Z_n)| > n$
(c) there exist a bounded sequence $\{Z_n\}$ such that $|f(Z_n)| > n$
(d) there exist a sequence $\{Z_n\}$ such that $Z_n \to 0$ and $f(Z_n) \to 2$.
Ans. (c).

125. Let $f(z)$ be an entire function such that for some constant α, $|f(z)| \leq \alpha |z|^3$ for $|z| \geq 1$ and $f(z) = f(iz)$, $\forall z \in \mathbb{C}$. Then, GATE(MA): 2006
(a) $f(z) = \alpha z^3$, $\forall z \in \mathbb{C}$ (b) $f(z)$ is constant
(c) $f(z)$ is quadratic polynomial (d) no such $f(z)$ exists.
Ans. (b). Since $f(z)$ is analytic and $|f(z)| \leq \alpha |z|^3$ so, $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$. Also $f(z) = f(iz) \Rightarrow a_1 = a_2 = a_3 = 0$. Therefore $f(z) = a_0$.

126. Let f be the entire function on C such that $f(z) \leq 100 \log |z|$ for each z with $|z| \geq 2$. If $f(i) = 2i$ then $f(1)$ must be GATE(MA): 2013
(a) 2 (b) $2i$ (c) i (d) Cannot be determined
Ans. (b)

127. Let $\mathbb{C}$ be the contour $|z| = 2$ oriented in the anti-clockwise direction. The value of the integral $\oint_C ze^{\frac{3}{z}} dz$ is GATE(MA): 2013
(a) $3\pi i$ (b) $5\pi i$ (c) $7\pi i$ (d) $9\pi i$
Ans. (d)

128. Let $f : \mathbb{C}\{3i\} \to \mathbb{C}$ be defined by $f(z) = \frac{z-i}{iz+3}$. Which of the following statement about f is false? GATE(MA): 2013
(a) f is conformal on C
(b) f maps circles $\mathbb{C}\{3i\}$ onto circles in C.
(c) All the fixed points of f are in the region $\{z \in C : Im(z) > 0\}$
(d) There is no straight line in $\mathbb{C}\{3i\}$ which is mapped onto a straight line in C by f.
Ans. (c)

129. The image of the region $\{z \in \mathbb{C} : Re(z) > Im(z) > 0\}$ under the mapping $z \mapsto e^{z^2}$ is
(a) $\{w \in C : Re(w) > 0, Im(w) > 0\}$ (b) $\{w \in C : Re(w) > 0, Im(w) > 0, |w| > 1\}$
(c) $\{w \in C : |w| > 1\}$ (d) $\{w \in C : Im(w) > 0, |w| > 1\}$ GATE(MA): 2013
Ans. (c)

130. Let f be an analytic function on $\bar{D} = \{z \in C : |z| \leq 1\}$. Assume that $|f(z)| \leq 1$ for each $z \in \bar{D}$. Then, which of the following is not a possible value of $(e^f)''(0)$? GATE(MA): 2013
(a) 2 (b) 6 (c) $\frac{7e^{\frac{1}{9}}}{9}$ (d) $\sqrt{2} + \sqrt{2}$.
Ans. (b). Since $(e^f)''(0) = e'(0)[f''(0) + f'(0)^2]$.

131. The coefficient of $(z - \pi)^2$ in the Taylor series expansion of

$$f(z) = \begin{cases} \frac{\sin z}{z-\pi} & \text{if } z \neq \pi \\ -1 & \text{if} z = \pi \end{cases}$$

around π is GATE(MA): 2013
(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $-\frac{1}{6}$
Ans. (c).

132. The function $f(z) = |z|^2 + i\bar{z} + 1$ is differentiable at GATE(MA): 2014
(a) i (b) 1 (c) $-i$ (d) no point in $\mathbb{C}$.
Ans. (c). Since $f(x, y) = x^2 + y^2 + i(x - iy) + 1$. check the Cauchy Riemann equations.

133. The radius of convergence of the power serious $\sum_{n=0}^{\infty} 4^{(-1)^n n} z^{2n}$ is GATE(MA): 2014
Ans. $R = \frac{1}{2}$. Since

$$a_n = \begin{cases} 0, & n = 2k - 1 \\ 4^n, & n = 2k, \qquad k = 1, 2, 3, \cdots \end{cases}$$

also $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{k\to\infty} |4^k|^{\frac{1}{2k}} = 2$.

134. The maximum modulus of e^{z^2} on the set $S = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1, 0 \leq Im(z) \leq 1\}$ is
(a) $\frac{2}{e}$ (b) e (c) $e + 1$ (d) e^2 GATE(MA): 2014
Ans. (b).

135. Let $\Omega = \{z \in \mathbb{C} : Im(z) > 0\}$ and let $\mathbb{C}$ be a smooth curve lying in Ω with initial point $-1 + 2i$ and final point $1 + 2i$. The value of $\int_C \frac{1+2z}{1+z} dz$ is GATE(MA): 2014
(a) $4 - \frac{1}{2}ln2 + i\frac{\pi}{4}$ (b) $-4 + \frac{1}{2}ln2 + i\frac{\pi}{4}$ (c) $4 + \frac{1}{2}ln2 - i\frac{\pi}{4}$ (d) $4 - \frac{1}{2}ln2 + i\frac{\pi}{4}$
Ans. (a)

136. If $a \in \mathbb{C}$ with $|a| < 1$, then the value of $\frac{(1-|a|^2)}{\pi} \int_\Gamma \frac{|dz|}{|z+a|^2}$, where Γ is the simple closed curve $|z| = 1$ taken with the positive orientation is GATE(MA): 2014
Ans. 1.99 to 2.1.

137. If the power series $\sum_{n=0}^{\infty} a_n(z + 3 - i)$ convergence at $5i$ and diverges at $-3i$, then the power series GATE(MA): 2014
(a) converges at $-2 + 5i$ and diverges at $2 - 3i$
(b) converges at $2 - 3i$ and diverges at $-2 + 5i$
(c) converges at both $2 - 3i$ and $-2 + 5i$

(d) diverges at both $2-3i$ and $-2+5i$
Ans. (a).

138. Let $u(x,y)=x^3+ax^y+bxy^2+2y^3$ be a harmonic function and $v(x,y)$ its harmonic conjugate. If $v(0,0)=1$, then $a+b+2v(1,1)$ is equal to GATE(MA): 2016
Ans. 9.9 to 10.1.

139. Let $\{\gamma = z \in \mathbb{C} : |z| = 2\}$ be oriented in the counter-clockwise direction. Let $I = \frac{1}{2\pi i}\oint_{\gamma} z^7\cos(\frac{1}{z^2})dz$. Then the value of I is equal to GATE(MA): 2016
Ans. 0.039 to 0.043.

140. Let (Z_n) be a sequence of distinct points in $D(0,1)=\{z\in\mathbb{C}:|z|<1\}$ with $\lim_{n\to\infty} z_n = 0$. Consider the following statements P and Q:
(P) : there exist a unique analytical function f on $D(0,1)$ such that $f(z_n)=\sin(z_n)$ for all z_n.
(Q) : there exist a unique analytical function f on $D(0,1)$ such that $f(z_n)=0$ if n is even and $f(z_n)=1$ if n is odd. GATE(MA): 2016
Which of the following statement hold TRUE?
(a) both P and Q (b) only P (c) only Q Neither P nor Q.
Ans. (b).

141. Consider the power series $\sum_{n=0}^{\infty} a_n z^n$ where $a_n = \begin{cases} \frac{1}{3^n}, & \text{if n is even} \\ \frac{1}{5^n}, & \text{if n is odd} \end{cases}$ The radius of convergence of the power serious is equal to GATE(MA): 2015
Ans. 3.

142. Let $C=\{z\in\mathbb{C}:|z-i|=2\}$. Then $\frac{1}{2\pi}\int_{\mathbb{C}}\frac{z^2-4}{z^2+4}dz$ is equal to GATE(MA): 2015
Ans. -2.

143. Let Let $D=\{z\in\mathbb{C}:|z|<1\}$. Then there exist a non-constant analytic function f on D such that for all $n=2,3,4,\cdots$ GATE(MA): 2015
(a) $f(\frac{\sqrt{-1}}{n})=0$ (b) $f(\frac{1}{n})=0$ (c) $f(1-\frac{1}{n})=0$ (d) $f(\frac{1}{2}-\frac{1}{n})=0$
Ans. c.

144. Let $\sum_{-\infty}^{\infty} a_n z^n$ be the Laurent series expansion of $f(z)=\frac{1}{2z^2-13z+15}$ in the annulus $\frac{3}{2}<|z|<5$. Then $\frac{a_1}{a_2}$ is equal to GATE(MA): 2015
Ans. 5.

145. The value of $\frac{i}{4-\pi}\int_{|z|=4}\frac{dz}{z\cos z}$ is equal to GATE(MA): 2015
Ans. 2.

Chapter 3

Numerical Solution of Differential Equations

3.1 Introduction

3.2 Multiple Choice Questions(MCQ)

1. A Runge-Kutta method for numerically solving the initial value ODE

$$y' = \frac{dy}{dx} = f(x,y) \quad \text{with} \quad y(x_0) = y_0$$

is given by (for h small)

$$y_1 = y(x_0 + h) = y_0 + k$$

$$\begin{aligned} \text{Where} \quad k &= \alpha k_1 + \beta k_2 \\ k_1 &= hf(x_0, y_0) \\ k_2 &= h[f(x_0 + mh, y_0 + nk_1)] \end{aligned}$$

The objective is to determine the constants α, β, m, n such that the above formula is accurate to order 2 (that is the error is $O(h^3)$). Which of the following are correct sets of values for these constants ? **NET(MS): (June)2013**
(a) $\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \ m = 1, n = 1$ (b) $\alpha = 2, \beta = 1, \ m = \frac{1}{2}, n = \frac{1}{2},$
(c) $\alpha = \frac{1}{3}, \beta = \frac{2}{3}, \ m = \frac{3}{4}, n = \frac{3}{4},$ (d) $\alpha = \frac{3}{4}, \beta = \frac{1}{4}, \ m = 2, n = 2$
Ans. (a), (c) and (d). (Note: The three answers are correct.)

2. Using Euler's method taking step size = 0.1, the approximate value of y obtained corresponding to $x = 0.2$ for the initial value problem $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$, is **GATE/MA-12**
(A) 1.322 (B)1.122 (C) 1.222 (D) 1.110
Ans. (c)

3. Runge Kutta method has a truncation error, which is of the order **CS-4, 6, 10**
(a) h^2 (b) h^3 (c) h^4 (d) none of these.
Ans. (b)

4. For $\frac{dy}{dx} = x+y$, and $y(0) = 1$, the the value of $y(1.1)$ according to the Euler method is [taking $h = 0.1$] **CS-312/08**
(a)0.1 (b) 0.3 (c) 1.1 (d)0.9
Ans. (c) $y_1(1.1) = y_0 + hf(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$.
5. The ordinary differential equations are solved numerically by?
(a) Euler method (b)Taylor method (c) Runge-Kutta method (d) All of these.
Ans. (d)
6. Consider the initial value problem $y' = x(y + x) - 2, y(0) = 2$. Use Euler's method with step sizes $h = 0.3$ to compute approximations to $y(0.6)$ is equals to
(a)0.953 (b)0.0953 (c) 0.909 (d)-0.953
Ans.(a) The Euler method applied to the given problem gives
$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, \cdots$. h = 0.3: $n = 0, x_0 = 0$ $y_1 = y_0 + 0.3[-2] = 2 - 0.6 = 1.4$. $n = 1, x_1 = 0.3$. $y_2 = y_1 + 0.3[0.3(y_1 + 0.3) - 2] = 1.4 - 0.447 = 0.953$.
7. The approximate eigenvalue of the matrix

$$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$$

obtained after two iterations of power method, with the initial vector $[1, 1, 1]^T$ is
(a)7.768 (b)9.468 (c) 10.548 (d)19.468 **GATE(MA): 2012**
Ans.(c)
8. Consider the system of equations

$$\begin{bmatrix} 5 & 2 & 1 \\ -2 & 5 & 2 \\ -1 & 2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13 \\ -22 \\ 14 \end{bmatrix}$$

with the initial guess of the solution $[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}]^T = [1, 1, 1]^T$, the approximate value of the solution $[x_1^{(1)}, x_2^{(1)}, x_3^{(1)}]^T$ after one iteration by the Gauss-Seidel method is
(a)$[2, -4.4, 1.625]^T$ (b)$[2, -4, -3]^T$ (c) $[2, 4.4, 1.625]^T$ (d)$[2, -4, 3]^T$ **GATE(MA): 2011**
Ans.(a)
9. Consider the system of equations

$$\begin{bmatrix} 5 & -1 & 1 \\ 2 & 4 & 0 \\ 1 & 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix}.$$

Using Jacobi's method with the initial guess of the solution $[x^{(0)}, y^{(0)}, z^{(0)}]^T = [2.0, 3.0, 0.0]^T$, the approximate value of the solution $[x^{(2)}, y^{(2)}, z^{(2)}]^T$ after two iteration, is
(a)$[2.64, -1.70, -1.12]^T$ (b)$[2.64, -1.70, 1.12]^T$
(c) $[2.64, 1.70, -1.12]^T$ (d)$[2.64, 1.70, 1.12]^T$ **GATE(MA): 2012**
Ans.(c)

Statement for the Linked Answer Questions 10 and 11.
The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ can be decomposed into the product of a lower triangular

matrix L and an upper triangular matrix U as $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & L_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$. Let $x, z \in \mathbb{R}^3$ and $b = [1,1,1]^T$

10. The solution $z = [z_1, z_2, z_3]^T$ of the system $Lz = b$ is
(a)$[-1,-1,-2]^T$ (b)$[1,-1,2]^T$ (c) $[1,-1,-2]^T$ (d)$[-1,1,2]^T$ **GATE(MA): 2011**
Ans.(c)

11. The solution $x = [x_1, x_2, x_3]^T$ of the system $Lx = z$ is
(a)$[2,1,-2]^T$ (b)$[2,1,2]^T$ (c) $[-2,-1,-2]^T$ (d)$[-2,1,-2]^T$ **GATE(MA): 2011**
Ans.(a)

Statement for the Linked Answer Questions 12 and 13.

For a differentiable function $f(x)$, the integral $\int_0^h f(x)dx$ is approximated by the formula $h[a_0 f(0) + a_1 f(h) + h^2[b_0 f(0) + b_1 f(h)]]$, which is exact for all polynomials of degree atmost 3.

12. The values of a_1 and b_1, respectively are
(a) $\frac{1}{2}$ and $\frac{-1}{12}$ (b) $\frac{-1}{12}$ and $\frac{1}{2}$ (c) $\frac{1}{2}$ and $\frac{1}{12}$ (d) $\frac{1}{12}$ and $\frac{1}{2}$ **GATE(MA): 2010**
Ans.(a)

13. The values of a_0 and b_0, respectively are
(a) $\frac{1}{2}$ and $\frac{1}{2}$ (b) $\frac{1}{12}$ and $\frac{-1}{12}$ (c) $\frac{1}{2}$ and $\frac{1}{12}$ (d) $\frac{1}{2}$ and $\frac{-1}{12}$ **GATE(MA): 2010**
Ans.(c)

Hint. We have $\int_0^h f(x)dx = h[a_0 f(0) + a_1 f(h) + h^2[b_0 f(0) + b_1 f(h)]$.
For $f(x) = 1$, $h = h(a_0 + a_1) \Rightarrow a_0 + a_1 = 1$.
For $f(x) = x$, $\frac{h^2}{2} = h(a_1 h) + h^2(b_0 + b_1) \Rightarrow 2a_1 + 2b_0 + 2b_1 = 1$.
For $f(x) = x^2$, $\frac{h^3}{3} = h(a_1 h^2) + h^2(2b_1 h) \Rightarrow 3a_1 + 6b_1 = 1$.
For $f(x) = x^3$, $\frac{h^4}{4} = h(a_1 h^3) + h^2(3b_1 h) \Rightarrow 4a_1 + 12b_1 = 1$.
Solving the above equations, we get $a_0 = \frac{1}{2}, b_0 = \frac{1}{12}, a_1 = \frac{1}{2}, b_1 = \frac{-1}{12}$.

14. Consider the Quadrature formula $\int_0^h f(x)dx = \{\alpha f(0) + \beta f(\frac{3h}{4}) + \gamma f(h)\}h$. The values of α, β and γ for which this is exact for polynomials of as high degree as possible are
(a) $\alpha = \frac{5}{18}, \beta = \frac{8}{9}, \gamma = -\frac{1}{6}$ (b) $\alpha = \frac{1}{2}, \beta = -\frac{1}{4}, \gamma = -\frac{3}{4}$ **GATE(MA): 2003**
(c) $\alpha = 0, \beta = 1, \gamma = -\frac{1}{4}$ (d) $\alpha = 1, \beta = 2, \gamma = 3$
Ans.(a)

15. If a quadrature formula $\frac{3}{2}f(-\frac{1}{3}) + Kf(\frac{1}{3}) + \frac{1}{2}f(1)$, that approximates $\int_{-1}^{1} f(x)dx$, is found to be exact for quadratic polynomials, then the value of K is
(a) 2 (b) 1 (c) 0 (d) -1. **GATE(MA): 2008**
Ans.(c)

Hint. We have $\int_{-1}^{1} f(x)dx = \frac{3}{2}f(-\frac{1}{3}) + Kf(\frac{1}{3}) + \frac{1}{2}f(1)$.
For $f(x) = 1$, $2 = \frac{3}{2}\cdot 1 + K\cdot 1 + \frac{1}{2}\cdot 1 \Rightarrow K = 0$.

16. The values of the constants α, β, x_1 for which the quadratic formula $\int_0^1 f(x)dx = \{\alpha f(0) + \beta f(x_1)$ is exact for polynomials of degree as high as possible are
(a) $\alpha = \frac{2}{3}, \beta = \frac{1}{4}, x_1 = \frac{3}{4}$ (b) $\alpha = \frac{3}{4}, \beta = \frac{1}{4}, x_1 = \frac{2}{3}$ **GATE(MA): 2005**
(c) $\alpha = \frac{1}{4}, \beta = \frac{3}{4}, x_1 = \frac{2}{3}$ (d) $\alpha = \frac{2}{3}, \beta = \frac{3}{4}, x_1 = \frac{1}{4}$
Ans.(c)

17. If $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & A \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & L_{32} & -53 \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$. Then the value of A is
(a) -2 (b) -1 (c) 1 (d) 2. **GATE(MA): 2008**
Ans.(a)

18. Using the least squares method, if a curve $y = ax^2 + bx + c$ is fitted to the collinear data points $(-1, -3), (1, 1), (3, 5)$ and $(7, 13)$, then the triplet (a, b, c) is equal to
(a) $(-1, 2, 0)$ (b) (0, 2, -1) (c) $(2, -1, 0)$ (d) $(0, -1, 2)$ **GATE(MA): 2008**
Ans.(a)

19. A quadratic polynomial $p(x)$ is constructed by interpolating the data points $(0, 1), (1, e)$ and $(2, e^2)$. If $\sqrt{e}$ is approximated by using $p(x)$, then its approximate value is **GATE(MA): 2008**
(a) $\frac{1}{8}(3 + 6e - e^2)$ (b) $\frac{1}{8}(3 - 6e + 2e^2)$ (c) $\frac{1}{8}(3 - 6e - e^2)$ (d) $\frac{1}{8}(3 + 6e - 2e^2)$.
Ans.(b)

20. If $y_{i+1} = y_i + h\phi(f, x_i, y_i, h)$, $i = 1, 2, \cdots$, where $\phi(f, x, y, h) = af(x, y) + bf[x+h, y+hf(x, y)]$, is a second order accurate scheme to solve the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, then a and b, respectively are **GATE(MA): 2008**
(a) $\frac{h}{2}, \frac{h}{2}$ (b) $1, -1$ (c) $\frac{1}{2}, \frac{1}{2}$ (d) $h, -h$.
Ans.(a)

21. The least approximation of first degree to the function $f(x) = \sin x$ over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is
(a) $\frac{24x}{\pi^3}$ (b) $\frac{24x}{\pi^2}$ (c) $\frac{24x}{\pi}$ (d) $24x$. **GATE(MA): 2001**
Ans.(b)

22. The order of the numerical differentiation formula $f'(x_0) = \frac{1}{12h^2}[-\{f(x_0 - 2h) + f(x_0 + 2h)\} + 16\{f(x_0 - h) + f(x_0 + h)\} - 30f(x_0)]$ is
(a) 2 (b) 3 (c) 4 (d) 1 **GATE(MA): 2001**
Ans.(a)
Hint. We have $f'(x_0) = \frac{1}{12h^2}[-\{f(x_0 - 2h) + f(x_0 + 2h)\} + 16\{f(x_0 - h) + f(x_0 + h)\} - 30f(x_0)] = \frac{1}{12h^2}[E^2 - 16E + 30 - 16E^{-1} + E^{-2}]f(x_0)$ which is of order 2.

23. The method $y_{n+1} = y_n + \frac{1}{4}(k_1 + 3k_2)$, $n = 0, 1, 2, \cdots$, where $k_1 = hf(x_n, y_n)$ and $k_2 = hf(x_n + \frac{2h}{3}, y_n + \frac{2k_1}{3})$ is used to solve the initial value problem $y' = f(x, y) = -10y$, $y(0) = 1$. The method will produce stable results, if the step size h satisfies **GATE(MA): 2001**

(a) $0.2 < h < 0.5$ (b) $0 < h < 0.5$ (c) $0 < h < 1$ (d) $0 < h < 0.2$
Ans.(d)
Hint. We have

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{4}(k_1 + 3k_2),\ n = 0, 1, 2, \cdots \\ &= y_n + \frac{1}{4}\{4h \cdot f(x_n, y_n) + 3h \cdot \frac{2h}{3} f_x(x_n, y_n) + 3h\frac{2k_1}{3} f_y(x_n, y_n) + \cdots\} \\ &= (1 - 10h)y_n + \frac{3h}{4} \cdot \frac{2h}{3} \cdot 0 + \frac{3h}{4} \cdot \frac{2k_1}{3} \cdot -10\} \\ &= (1 - 10h)y_n + \frac{600h^2 y_n}{12} \\ &= (1 - 10h + 50h^2)y_n \end{aligned}$$

Therefore, the method will produce stable results, if the step size h satisfies $|1 - 10h + 50h^2| < 1$. So here (d) i.e. $0 < h < 0.2$ is the result.

24. The Runge-Kutta method of order four is used to solve the differential equation $\frac{dy}{dx} = f(x)$, $y(0) = 0$ with step size h. The solution at $x = h$ is given by
(a) $y(h) = \frac{h}{6}\left[f(0) + 4f(\frac{h}{2}) + f(h)\right]$ (b) $y(h) = \frac{h}{6}\left[f(0) + 2f(\frac{h}{2}) + f(h)\right]$
(c) $y(h) = \frac{h}{6}\left[f(0) + f(h)\right]$ (d)$y(h) = \frac{h}{6}\left[2f(0) + f(\frac{h}{2}) + 2f(h)\right]$ **GATE(MA): 2005**
Ans.(a)
Hint. $y(h) = y(0) + \frac{k_1+2k_2+2k_3+k_4}{6}$ where $k_1 = hf(x_0, y_0) = hf(0)$, $k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = hf(\frac{h}{2})$, $k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = hf(\frac{h}{2})$, $k_4 = hf(x_0 + h, y_0 + k_3) = hf(h)$.

25. The fourth divided difference of the polynomial $3x^3 + 11x^2 + 5x + 11$ over the points $x = 0, 1, 4, 6$ and 7 is
(a) 18 (b) 11 (c) 3 (d) 0 **GATE(MA): 2002**
Ans.(d)

26. The polynomial of the least degree interpolating the data $(0, 4), (1, 5), (2, 8)$ and $(3, 13)$ is
(a) 4 (b) 3 (c) 2 (d) 1 **GATE(MA): 2002**
Ans.(c)

27. For the matrix $\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$, the bound for the eigenvalues predicted by Gershgorin's theorem is
(a) 3 (b) 1 (c) 2 (d) 4 **GATE(MA): 2002**
Ans.(b)
Ans. The bounds for the eigenvalues predicted by Gershgorin's theorem is as follows. For i, $i = 1, 2, 3, 4$-th row, $|\lambda - a_{ii}| \le |a_{i1}| + |a_{i2}| + |a_{i3}| + |a_{i4}| - |a_{ii}|$. Therefore, $|\lambda| \le 3$ for $i = 1, 24$ and $|\lambda| \le 1$ for $i = 3$. So the bound for the eigenvalue is $|\lambda| \le 1$.

28. Determine the LU decomposition of the matrix $A = \begin{bmatrix} 5 & -2 & -3 \\ 20 & -5 & -13 \\ 35 & -5 & -17 \end{bmatrix}$ with L having all its diagonal entries 1 and hence solve the system $AX = [0, 2, 13]^T$.
(a) $x_1 = 1, x_2 = 2, x_3 = 0$, (b) $x_1 = x_2 = 0, x_3 = 1$,

(c) $x_1 = 1 = x_2 = x_3$, (d) $x_1 = x_2 = 0, x_3 =$ arbitrary. **GATE(MA): 2002**
Ans.(d)

29. Using the Runge-Kutta method of the order 4 and taking the step size $h = 0.1$, determine $y(0.1)$, where $y(x)$ is the solution of $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 1$.
(a) 0 (b) 1 (c) 2 (d) 3 **GATE(MA): 2002**
Ans.(a)

30. A lower bound on the polynomial interpolation error $e_2(\bar{x})$ for $f(x) = \ln(x)$, with $x_0 = 2$, $x_1 = 2$, $x_2 = 4$ and $\bar{x} = \frac{5}{4}$ is given by
(a) $\frac{1}{256}$ (b) $\frac{1}{64}$ (c) $\frac{1}{512}$ (d) 0
Ans.(c)

31. Let $l_k(x)$, $k = 0, 1, \cdots, n$ denote the Lagrange's polynomials of degree n for the nodes $x_0, x_1, \cdots, x_n$. Then the value of $\sum\limits_{k=0}^{n} l_k(x)$ is
(a) 0 (b) 1 (c) $x^n + 1$ (d)$x^n - 1$ **GATE(MA): 2010**
Ans.(b)
Hint. $\sum\limits_{k=0}^{n} l_k(x) = 1$ and $l_i(x_j) = \delta_i^j$.

32. If $f(x)$ has an isolated zero of multiplicity 3 at $x = \xi$ and the iteration $x_{n+1} = x_n - \frac{3f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \cdots$ converges to ξ, then the rate of converges is **GATE(MA): 2002**
(a) Linear (b) faster than linear but slower than quadratic (c) quadratic (d) cubic
Ans.(a)

33. The best possible error estimate in the Gauss-Hermite formula with 3 points for calculating the integral $\int\limits_{-\infty}^{\infty} x^4 e^{-x^2} dx$ is
(a) 0 (b) 0.30 (c) 0.65 (d) 1.20 **GATE(MA): 2002**
Ans.(c)

34. Let $\phi : [0, 1] \to \mathbb{R}$ be the three times continuously differentiable. Suppose that the iterates defined by $x_{n+1} = \phi(x_n)$, $n \leq 0$ converge to the fixed point ξ of ϕ. If the order of convergence is three, then
(a) $\phi'(\xi) = 0$, $\phi''(\xi) = 0$ (b) $\phi'(\xi) \neq 0$, $\phi''(\xi) = 0$ (c) $\phi'(\xi) = 0$, $\phi''(\xi) \neq 0$ (d) $\phi'(\xi) \neq 0$, $\phi''(\xi) \neq 0$ **GATE(MA): 2009**
Ans.(a)

35. Let $f : [0, 2] \to \mathbb{R}$ be a twice continuously differentiable function. If $\int\limits_{0}^{2} f(x)dx = 2f(1)$, then the error in the approximation is
(a) $\frac{f'(\xi)}{12}$ for some $\xi \in (0, 2)$ (b) $\frac{f'(\xi)}{2}$ for some $\xi \in (0, 2)$ **GATE(MA): 2009**
(c) $\frac{f'(\xi)}{3}$ for some $\xi \in (0, 2)$ (d) $\frac{f'(\xi)}{6}$ for some $\xi \in (0, 2)$
Ans.(d)

36. Let $f : [0, 4] \to \mathbb{R}$ be a three times continuously differentiable function. Then the value of $f[1, 2, 3, 4]$ is
(a) $\frac{f'(\xi)}{3}$ for some $\xi \in (0, 4)$ (b) $\frac{f'(\xi)}{6}$ for some $\xi \in (0, 4)$ **GATE(MA): 2009**

(c) $\frac{f''(\xi)}{3}$ for some $\xi \in (0,4)$ (d) $\frac{f''(\xi)}{6}$ for some $\xi \in (0,4)$
Ans.(d)

37. Consider the quadrature formula $\int_{-1}^{1} |x| f(x)dx \approx \frac{1}{2}[f(x_0)+f(x_1)]$ where x_0 and x_1 are quadrature points. Then the highest degree of the polynomial, for which the above formula is exact equals
(a) 1 (b) 2 (c) 3 (d) 4 **GATE(MA): 2007**
Ans.(a)
Hint. Since the quadrature formula is exact for $f(x) = 1$ and $f(x) = x$ but not exact for $f(x) = x^2$.

38. Suppose that x_0 is sufficiently close to 3. Which of the following iterations $x_{n+1} = g(x_n)$ will converge to the fixed point $x = 3$?
(a) $x_{n+1} = -16 + 6x_n + \frac{3}{x_n}$ (b) $x_{n+1} = \sqrt{3 + x_n}$ **GATE(MA): 2007**
(b) $x_{n+1} = \frac{3}{x_n - 2}$ (d) $x_{n+1} = \frac{x_n^2 - 3}{2}$
Ans.(b)
Hint. Let $x_n \to x$, then $x^2 = 3 + 2x$ satisfies by $x = 3$ and $|\phi'(x)| = |\frac{1}{\sqrt{3+2x}}| < 1$.

39. Suppose the iterates x_n generated by $x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$ converges to a double zero $x = a$ of $f(x)$. Then the convergence has order **GATE(MA): 2005**
(a) 1 (b) 2 (c) 3 (d) 1.6.
Ans.(b)

40. Consider the initial value problem(IVP) $\frac{dy}{dx} = f(x, y(x))$, $y(x_0) = y_0$. Let $y_1 = y_0 + w_1 k_1 + 3k_2$ approximate the solution of the above IVP at $x_1 = x_0 + h$ with $k_1 = hf(x_0, y_0)$, $k_2 = hf(x_0 + \frac{h}{6}, y_0 + \frac{k_1}{6})$ and h being the step size. If the formula for y_1 yield a second order method then the value of w_1 is
(a) -1 (b) -2 (c) 3 (d) $\frac{1}{6}$ **GATE(MA): 2006**
Ans.(d)

41. In solving the ordinary differential equation $y' = 2x$, $y(0) = 0$ using Euler's method, the iterates y_n, $n \in N$ satisfy **GATE(MA): 2004**
(a) $y_n = x_n^2$ (b) $y_n = 2x_n$ (c) $y_n = x_n x_{n-1}$ (d) $y_n = x_{n-1} + x_n$.
Ans.(c)

42. To find the positive square root of $a > 0$ by solving $x^2 - a = 0$ by the Newton-Raphson method, if x_n denotes the nth iterate with $x_0 > 0\ x \neq \sqrt{a}$, then the sequence $\{x_n, n \geq 1\}$ is
(a) strictly decreasing (b)strictly increasing (c) constant (d) not convergent
Ans.(b) **GATE(MA): 2004**

43. For what values of α and β, the quadrature formula $\int_{-1}^{1} f(x)dx = \alpha f(-1) + f(\beta)$ is exact for all polynomials of degree≤ 1? **GATE(MA): 2009**
(a) $\alpha = 1, \beta = 1$ (b) $\alpha = -1, \beta = 1$ (c) $\alpha = 1, \beta = -1$ (d) $\alpha = -1, \beta = -1$
Ans.(a)

44. An iterative method to find the n-th root ($n \in N$) of a positive number **a** is given by $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k^{n-1}}\right)$. A value of n for which this iterative method fails to converge is

(a) 1 (b) 2 (c) 3 (d) 8 **GATE(MA): 2005**
Ans.(a)
Hint. For n=1, the $\{x_k\}$ is monotonic increasing, so it is not convergent.

45. Suppose the function $u(x)$ interpolates $f(x)$ at $x_0, x_1, x_2, \cdots, x_{n-1}$ and the function $v(x)$ interpolates $f(x)$ at $x_1, x_2, \cdots, x_{n-1}, x_n$. Then a function $F(x)$ which interpolates $f(x)$ all the points $x_0, x_1, x_2, \cdots, x_{n-1}, x_n$ is given by
(a) $F(x) = \frac{(x_n-x)u(x)-(x-x_0)v(x)}{x_n-x_0}$ (b) $F(x) = \frac{(x_n-x)u(x)+(x-x_0)v(x)}{x_n-x_0}$ **GATE(MA): 2005**
(c) $F(x) = \frac{(x_n-x)v(x)+(x-x_0)u(x)}{x_n-x_0}$ (d) $F(x) = \frac{(x_n-x)v(x)-(x-x_0)u(x)}{x_n-x_0}$
Ans.(d)

46. An iterative scheme is given by $x_{n+1} = \frac{1}{5}\left(16 - \frac{12}{x_n}\right)$, $n \in N \bigsqcup \{0\}$. Such a scheme with suitable x_0 will **GATE(MA): 2004**
(a) not converge (b) Converge to 1.6 (c) Converge to 1.8 (d) Converge to 2
Ans.(d)

47. Let M be the length of the initial interval $[a_0, b_0]$ containing a solution of $f(x) = 0$. Let $[x_0, x_1, x_2 \cdots]$ represent the successive points generated by the bisection method. Then the minimum number of iterations required to guarantee an approximation to the solution with an accuracy of ϵ is given by **GATE(MA): 2003**
(a) $-2 - \frac{\log(\frac{\epsilon}{M})}{\log 2}$ (b) $-2 + \frac{\log(\frac{\epsilon}{M})}{\log 2}$ (c) $-2 + \frac{\log(M\epsilon)}{\log 2}$ (d) (a) $-2 - \frac{\log(\frac{\epsilon}{M})}{(\log 2)^2}$
Ans.(a)

48. Suppose the matrix $M = \begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$ has a unique Cholesky decomposition of the form $M = LL^T$, where L is a lower triangular matrix. The range of the values of α is
(a) $-2 < \alpha < 2$ (b) $\alpha > 2$ (c) $-2 < \alpha < \frac{3}{2}$ (d) $\frac{3}{2} < \alpha < 2$ **GATE(MA): 2005**
Ans.(c)

49. The smallest degree of the polynomial that interpolates the data

x	-2	-1	0	1	2	3
$f(x)$	-58	-21	-12	-13	-6	27

is **GATE(MA): 2005**
(a) 3 (b) 4 (c) 5 (d) 6
Ans.(a)
Hint. Since $\Delta^3 f(x)$ is constant. Hence polynomial must be of degree 3.

50. Let the following discrete data be obtained from a curve $y = y(x)$:

$x:$	0	0.25	0.5	0.75	1.0
$y:$	1.0	0.9896	0.9589	0.9089	0.8415

Let S be the solid of revolution obtained by rotating above curve about the x-axis between $x = 0$ and $x = 1$ and let V denote its volume, the approximate value of V, obtained using Simpson's $\frac{1}{3}$ rule $--------$. **GATE(MA): 2014**
Solution: Here $h = 0.25$, $y_0 = 1.0$, $y_1 = .9589$, $y_2 = 0.9089$, $y_4 = 0.8415$.. Then the required volution of the solid generated $V = \int_0^1 \pi y^2 dx = \frac{\pi h}{3}\left[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2\right] = \frac{0.25\pi}{3}\left[(1 + (0.8415)^2) + 4\left((0.9896)^2 + (0.9089)^2\right) + 2(0.9589)^2\right] = 0.2618 \times 10.77 = 2.82.$

51. Find the value of p such that the integration method $\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}\Big[f(x_0) + f(x_1)\Big] + ph^3\Big[f''(x_0) + f''(x_1)\Big]$ where $x_1 = x_0 + h$ provides exact result for highest degree polynomial. Find also the order of the method and the error term. **GATE(MA): 2001**
Ans. $p = -\frac{1}{24}$

52. The smallest value of $x(|x| < 1)$ correct to two decimal places satisfying the equation $x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \cdots = 0.4431135$ is **GATE(MA): 2000**
(a) 0.58 (b) 0.47 (c) 0.44 (d) 0.88
Ans.(b)

53. The Jacobi's iteration method for the set of equations
$x_1 + ax_2 = 2 \qquad 2ax_1 + x_2 = 7, \qquad (a \neq \frac{1}{\sqrt{2}})$converges for
(a) all values of a (b) $a = 1$ (c) $|a| < \frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}} < a < \sqrt{\frac{3}{2}}$ **GATE(MA): 2000**
Ans.(c)

54. The interpolating polynomial of highest degree which corresponds the functional values $f(-1) = 9,\ f(0) = 5,\ f(2) = 3,\ f(5) = 15$ is **GATE(MA): 2000**
(a) $x^3 + x^2 + 2x + 5$ (b) $x^2 - 3x + 5$ (c) $x^4 + 4x^3 + 5x^2 + 5$ (d) $x + 5$
Ans.(b)

55. Let the integral $I = \int_0^4 f(x)dx$ where $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 4 - x, & 2 \le x \le 4 \end{cases}$ Consider the following statements P and Q:
(P): If I_2 is the value of the integral obtained by the composite trapezoidal rule with two equal sub-intervals, then I_2 is exact.
(Q): If I_3 is the value of the integral obtained by the composite trapezoidal rule with three equal sub-intervals, then I_3 is exact.
Which of the above statements hold TRUE? **GATE(MA): 2016**
(a) both P and Q (b) only P (c) only Q (d) Neither P nor Q.
Ans.(b)
Hint. Here $f(x)$ is linear with two sub-intervals.

56. For the fixed point iteration $x_{k+1} = g(x_k),\ k = 0, 1, 2, \cdots$ consider the following statements P and Q:
(P): If $g(x) = 1 + \frac{2}{x}$, then the fixed point iteration converges to 2 for all $x_0 \in [1,\ 100]$.
(Q): If $g(x) = \sqrt{2 + x}$ then the fixed point iteration converges to 2 for all$x_0 \in [0,\ 100]$.
Which of the above statements hold TRUE? **GATE(MA): 2016**
(a) both P and Q (b) only P (c) only Q (d) Neither P nor Q.
Ans.(a)
Hint. Here $|g'(x)| < 1$ in both the cases.

57. Let J be the Jacobi iteration matrix of the linear system $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
Consider the following statements
(P): One of the eigenvalues of J lies in the interval $[2, 3]$.
(Q): The Jacobi iteration converges for the above system.

Which of the above statements hold TRUE? **GATE(MA): 2017**
(a) both P and Q (b) only P (c) only Q (d) Neither P nor Q.
Ans.(b)
Hint. Here $|g'(x)| < 1$ in both the cases.

58. Using the Gauss-Seidel iteration method with the initial guess $\{x_1^{(0)} = 3,\ x_2^{(0)} = 2.25,\ x_3^{(0)} = 1.625\}$, the second approximation $\{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\}$ for the solution to the system of equations

$$2x_1 - x_2 = 7$$
$$-x_1 + 2x_2 - x_3 = 1 \textbf{GATE(MA) : 2014}$$
$$-x_2 + 2x_3 = 1$$

is (a) $x_1^{(2)} = 5.3125,\ x_2^{(2)} = 4.4491,\ x_3^{(2)} = 2.1563$
(b) $x_1^{(2)} = 5.3125,\ x_2^{(2)} = 4.3125,\ x_3^{(2)} = 2.6563$
(c) $x_1^{(2)} = 5.3125,\ x_2^{(2)} = 4.4491,\ x_3^{(2)} = 2.6563$
(d) $x_1^{(2)} = 5.4991,\ x_2^{(2)} = 4.4491,\ x_3^{(2)} = 2.1563$
Ans.(b)

59. The third order divided difference of the function $f(x) = \frac{1}{x}$ with arguments a, b, c, d is
(a) $\frac{1}{abcd}$ (b) $-\frac{1}{abd}$ (c) $-\frac{1}{abcd}$ (d) $-\frac{1}{abc}$ **SET(MA): 2014**
Ans.(c)

60. If a points $x_1, x_2, x_3, \cdots, x_n$ are distinct, then for arbitrary real values $y_1, y_2, y_3, \cdots, y_n$ the degree of the unique interpolating polynomial $p(x_i) = y_i (1 \le i \le n)$ is **NET(MA): 2013**
(a) n (b) n-1 (c) $\le n - 1$ (d) $\le n$
Ans.(c)

61. The value of α and β such that $x_{n+1} = \alpha x_n(3 - \frac{x_n^2}{a}) + \beta(1 + \frac{a}{x_n^2})$ has 3^{rd} ordered convergence to $\sqrt{a}$ are **NET(DEC): 2016**
(a) $\alpha = \frac{3}{8}, \beta = \frac{1}{8}$ (b) $\alpha = \frac{1}{8}, \beta = \frac{3}{8}$ (c) $\alpha = \frac{2}{8}, \beta = \frac{2}{8}$ (d) $\alpha = \frac{1}{4}, \beta = \frac{3}{4}$
Ans.(b)

62. The maximum step size h such that the error in linear interpolation for the function $y = \sin x$ in $[0, \pi]$ is less than 5×10^{-5} is **GATE: 1999**
(a) 0.02 (b) 0.0002 (c) 0.04 (d) 0.06
Ans.(a)

63. The second order Runge-Kutta method is applied to the initial value problem $y' = -y,\ y(0) = y_0$, with step size h. Then $y(h)$ is **GATE: 1999**
(a) $y_0(h-1)^2$ (b) $\frac{y_0}{2}(h^2 - 2h + 2)$ (c) $\frac{y_0}{6}(h^2 - 2h + 2)$ (d) $y_0(1 - h + \frac{h^2}{2} + \frac{h^3}{6})$
Ans.(b)

64. Consider the iterative $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),\ n \ge 0$ for a given $x \ne 0$. Then
(a) x_n converges to $\sqrt{2}$ with rate of convergence 1. (b) x_n converges to $\sqrt{2}$ with rate of convergence 2.
(c) The given iteration is the fixed point iteration for $f(x) = x^2 - 2$ (d) The given iteration is the Newton's method for $f(x) = x^2 - 2$
Ans.(b) and (d) **NET(Jun): 2014**

65. The iterative scheme $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),\ n \geq 0$ for a given $x \neq 0$ is an instance of
(a) fixed point iteration for $f(x) = x^2 - 2$ (b)Newton's method for $f(x) = x^2 - 2$
(c) fixed point iteration for $f(x) = \frac{x^2+2}{2x}$ (d) Newton's method for $f(x) = x^2 + 2$
Ans.(b) and (c) **NET(DEC): 2015**

66. The iterative scheme $x_{n+1} = \frac{x_n}{2}\left(1 + \frac{a}{x_n^2}\right)$ converges to $\sqrt{a}$. The convergence is
(a) linear (b) quadratic (c) cubic (d) bi-quadratic **GATE(MA): 1999**
Ans.(b)

67. Let $f(x) = \sqrt{x+3}$ for $x \geq -3$. Consider the iteration $x_{n+1} = f(x_n)$, $x_0 = 0$, $n \leq 0$. The possible limits of the iteration are
(a) -1 (b) 3 (c) 0 (d) $\sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}}$ **NET(Dec): 2016**
Ans.(d)

68. Consider the function $f(x) = \sqrt{x+2}$ for $x \geq -2$ and the iteration $x_{n+1} = f(x_n)$, $x_0 = 1$, $n \leq 0$. What are the possible limits of the iteration ?
(a) $\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$ (b) -1 (c) 2 (d) 1 **NET(Jun): 2014**
Ans.(a) and (c).

69. The following numerical integration formula is exact for all polynomials of degree less than or equal to 3
(a) Trapezoidal (b) Simpson's $\frac{1}{3}$rd rule **NET(JUN): 2015**
(c)Simpson's $\frac{3}{8}$th rule (d) Gauss-Legendre 2 point formula.
Ans.(b), (c) and (d).

70. Consider the Runge-Kutta method of the form

$$\begin{aligned} y_{n+1} &= y_n + ak_1 + bk_2 \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \alpha h, y_n + \beta k_1) \end{aligned}$$

to approximate the solution of the initial value problem

$$y' = \frac{dy}{dx} = f(x, y(x)),\ y(x_0) = y_0.$$ **NET(MS) : (June)2016**

Which of the following choices of a, b, α, β yield a second order method?
(a) $a = \frac{1}{2},\ b = \frac{1}{2},\ \alpha = 1,\ \beta = 1$ (b) $a = 1,\ b = 1,\ \alpha = \frac{1}{2},\ \beta = \frac{1}{2}$
(c) $a = \frac{1}{4},\ b = \frac{3}{4},\ \alpha = \frac{2}{3},\ \beta = \frac{2}{3}$ (d) $a = \frac{3}{4},\ b = \frac{1}{4},\ \alpha = 1,\ \beta = 1$
Ans. (a) and (c).

71. Let $f : [0, 3] \to \mathbb{R}$ be defined by $f(x) = |1 - |x - 2||$ where $|\cdot|$ denotes the absolute value. Then the numerical approximation of $\int_0^3 f(x)dx$, which of the following statements are true? **NET(MS): (June)2016**
(a) The composite trapezoid rule with three equal subintervals is exact.
(b) The composite midpoint rule with three equals subintervals is exact.
(c) The composite trapezoid rule with four equal subintervals is exact.

(d) The composite midpoint rule with four equal subintervals is exact.
Ans. (a) and (b).

72. The fourth order Runge-Kutta method given by $u_{j+1} = u_j + \frac{k}{2}\Big[K_1 + 2K_2 + 2K_3 + K_4\Big],\ j = 0, 1, 2, \cdots$, is used to solve the initial value problem $\frac{du}{dt} = u$, $u(0) = \alpha$. If $u(1) = 1$ is obtained by taking the step size $h = 1$, then the valued of K_4 is ———. **GATE(MA): 2014**
Solution: Given initial value problem is $\frac{du}{dt} = u$, $u(0) = \alpha$ and $h = 1$ i.e., $f(t, u) = u$ and $t_0 = 0$, $u(0) = \alpha$.

$$
\begin{aligned}
& k_1 = hf(t_0, u_0) = u(0) = \alpha, \\
& k_2 = hf(t_0 + \frac{1}{2}, u_0 + \frac{k_1}{2}) = 1 \cdot f(\frac{1}{2}, \alpha + \frac{\alpha}{2}) = f(\frac{1}{2}, \frac{3\alpha}{2}). \\
& \Rightarrow k_2 = \frac{3\alpha}{2}, \\
& k_3 = hf(t_0 + \frac{1}{2}, u_0 + \frac{k_2}{2}) = 1 \cdot f(\frac{1}{2}, \alpha + \frac{3\alpha}{4}) = f(\frac{1}{2}, \frac{7\alpha}{4}) \\
& \Rightarrow k_3 = \frac{7\alpha}{4}, \\
& k_4 = hf(t_0 + h, u_0 + k_3) = 1 \cdot f(1, \alpha + \frac{7\alpha}{4}) = f(1, \frac{11\alpha}{4}). \\
& \Rightarrow k_4 = \frac{11\alpha}{4} \\
\because \quad & u_1 = u_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = \alpha + \frac{1}{6}[\alpha + 3\alpha + \frac{7\alpha}{2} + \frac{11}{4}\alpha] \\
& 1 = \frac{65}{24}\alpha \Rightarrow \alpha = 0.37 \\
& \therefore\ k_4 = \frac{11}{4} \times 0.37 = 1.02
\end{aligned}
$$

73. Let the polynomial x^4 be approximated by a polynomial of degree ≤ 2, which interpolates x^4 at $x = -1$, 0 and 1. Then, the maximum absolute interpolation over the interval $[-1,\ 1]$ is equal to ———. **GATE(MA): 2016**
Solution: The maximum error in quadratic interpolation for three equally spaced points is given by $|f(x) - p(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$ where $|f'''(x)| \leq M_3$, $\forall x \in [x_0,\ x_2]$
$\Rightarrow 24 \cdot x \leq M_3$, $\forall x \in [-1,\ 1] \Rightarrow M_3 = 24$
Therefore, Error $\leq \frac{1^3 \cdot 24}{9\sqrt{3}} = 1.539$
Thus, the maximum interpolation error over the interval $[-1, 1]$ is equal to 1.539.

74. Solve $\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y,\ y(0) = 3,\ y'(0) = 0$ to approximate $y(0.1)$ by using fourth order Range-Kutta method. **GATE(MA): 2000**
Solution: Substituting $\frac{dy}{dx} = z = f(x, y, z)$, then equation becomes $\frac{d^2y}{dx^2} = xz - y = g(x, y, z)$.

The initial conditions are $x = 0,\ y = 3,\ z = 0$. Also $h = 0.1$.

So, $k_1 = hf(x, y, z) = hz = 0.1 \times 0.0 = 0$

$$k_2 = hf(x + \frac{h}{2}, y + \frac{k_1}{2}, z + \frac{m_1}{2}) = h(z + \frac{m_1}{2}) = 0.1(0 - 0.15) = -0.015$$

$$k_3 = hf(x + \frac{h}{2}, y + \frac{k_2}{2}, z + \frac{m_2}{2}) = h(z + \frac{m_2}{2}) = 0.1(-\frac{0.30075}{2}) = -0.00150375$$

$$k_4 = hf(x + h, y + k_3, z + m_3) = h(z + m_3) = 0.1(-0.298498125) = -0.0298498125$$

$$m_1 = hg(x, y, z) = h(xz - y) = 0.1(0 - 3) = -0.3$$

$$m_2 = hg(x + \frac{h}{2}, y + \frac{k_1}{2}, z + \frac{m_1}{2}) = h\Big[(x + \frac{h}{2})(y + \frac{k_1}{2}) - (z + \frac{m_1}{2})\Big] = -0.30075$$

$$m_3 == hg(x + \frac{h}{2}, y + \frac{k_2}{2}, z + \frac{m_2}{2}) = h\Big[(x + \frac{h}{2})(y + \frac{k_2}{2}) - (z + \frac{m_2}{2})\Big] = -0.298498125$$

$$m_4 = hg(x + h, y + k_3, z + m_3) = h\Big[(x + h)(y + k_3) - (z + m_3)\Big] = -0.3014812312$$

$$\therefore\ y(0.1) = y(0) + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 3 + \frac{1}{6}\Big[0 + 2 \times (-0.015) + 2 \times (-0.0150375) + (-0.0298498)\Big] = 2.9887625$$

$$\text{So}, z(0.1) = z(0) + \frac{1}{6}[m_1 + 2m_2 + 2m_3 + m_4]$$

$$= 0 + \frac{1}{6}\Big[-0.3 + 2 \times (-0.30075) + 2 \times (-0.2984981) + (-0.301481)\Big] = -0.300163$$

75. Using the fourth order Range-Kutta method and taking the step size $h = 0.1$, determine $y(0.1)$, where $y(x)$ is the solution of $\frac{dy}{dx} + 2xy^2 = 0,\ y(0) = 1$. **GATE(MA): 2002**
Hint. Substituting $\frac{dy}{dx} = f(x, y) = -2xy^2$ with initial conditions $y(0) = 1$. Also $h = 0.1$.

76. Using the Newton-Raphson method with the initial guess $x^{(0)} = 6$, the approximate value of the real root of $x \log_{10} x = 4.77$, after the second iteration is ——. **GATE(MA): 2014**
Solution: Given $x_0 = 6,\ f(x) = x \log_{10} x - 4.77$ so, $f'(x) = 1 + \log x$. Then by Newton Raphson method, we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Now,
$$x_1 = 6 - \frac{6 \log 6 - 4.77}{1 + \log 6}$$
$$= 6 - \frac{6 \times 0.778 - 4.77}{1 + 0.778} = 6 - \frac{4.67 - 4.77}{1.778} = 6.056$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 6.056 - \frac{4.67 - 4.77}{1.778} = 6.07.$$

77. Let $\alpha \in \mathbb{R}$. If αx is the polynomial which interpolates the function $f(x) = \sin \pi x$ on $[-1, 1]$ at all the zeroes of the polynomial $4x^3 - 3x$, then α is ——. **GATE(MA): 2014**
Ans. The values of α lies between 0.46 to 0.48.

78. Let $p(x)$ be the polynomial of degree at most 3, that passes through the points $(-2, 12), (-1, 1), (0, 2)$ and $(2, -8)$. Then the coefficient of x^3 in $p(x)$ is equal to ——? **GATE(MA): 2015**
Ans.

79. Suppose that the Newton-Raphson method is applied to the equation $2x^2 + 1 - e^{x^2} = 0$ with an initial approximation x_0 sufficiently close to zero. Then for the root $x = 0$, the order of convergence of the method is equation to——–? **GATE(MA): 2015**
Ans.

80. If the trapezoidal rule with single interval $[0, 1]$ is exact for approximating the integral $\int_0^1 (x^3 - cx^2)dx$, then the value of c is equal to——–? **GATE(MA): 2015**
Ans.

81. If for some $\alpha, \beta \in \mathbb{R}$, the integration formula $\int_0^2 p(x)dx = p(\alpha)+p(\beta)$ holds for all polynomials $p(x)$ of degree at most 3, then the value of $3(\alpha - \beta)^2$ is equal to ——? **GATE(MA): 2015**
Ans.

82. Determine the LU decomposition of the matrix $A = \begin{bmatrix} 5 & -2 & -3 \\ 20 & -5 & -13 \\ 35 & -5 & -17 \end{bmatrix}$ with L having all its diagonal entries 1 and hence solve the system $AX = [0\ 2\ 13]^t$. **GATE(MA): 2002**
Ans.

83. Consider the (Cholesky's) algorithm given below for LL^T decomposition of a Symmetric Positive.
(i) Definite matrix A:
(ii) Compute $L_{11} = A_{11}^{\frac{1}{2}}$ **GATE(MA): 2003**
(iii) For $i = 2$ to N, Compute $L_{i,1} = \frac{A_{i,1}}{L_{i,1}}$
(iv) For $i = 2$ to N, Compute $L_{j,j} = \left(A_{j,j} - \sum_{m=1}^{j-1} L_{j,m}^2\right)^{\frac{1}{2}}$
Right alternative for filling the shaded box to complete the above algorithm is
(1) For $i = j + 1$ to N, Compute $L_{j,j} = \frac{1}{L_{j,j}}\left(A_{i,j} - \sum_{m=1}^{j-1} L_{i,m}L_{j,m}\right)$
(2) For $i = j$ to N, Compute $L_{i,j} = \frac{1}{L_{j,j}}\left(A_{i,j} - \sum_{m=1}^{j-1} L_{i,m}L_{j,m}\right)$
(3) For $i = j$ to N, Compute $L_{i,j} = \frac{1}{L_{j,j}}\left(A_{i,j} - \sum_{m=1}^{j-1} L_{i,m}L_{j,m}\right)$
(4) For $i = j + 1$ to N, Compute $L_{i,j} = \frac{1}{L_{j,j}}\left(A_{i,j} - \sum_{m=1}^{j-1} L_{i,m}^2\right)$
Ans. (4)

84. If the fourth order divided difference of $f(x) = \alpha x^4 + 5x^3 + 3x + 2$, $\alpha \in \mathbb{R}$, at the points $0.1, 0.2, 0.3, 0.4, 0.5$ is 5, then α equals ——? **GATE(MA): 2017**
Solution: Fourth order divided difference of $f(x)$ is

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{1}{4!} f^{iv}(0.1) = 5 \Rightarrow \frac{4!\alpha}{4!} = 5 \Rightarrow \alpha = 5$$

85. If the quadrature rule $\int_0^2 f(x)dx \approx c_1 f(0)+3f(c_2)$, where $c_1, c_2 \in \mathbb{R}$, is exact for all polynomials of degree ≤ 1, then $c_1 + 3c_2$ equals ——-? **GATE(MA): 2017**

Solution: Making the method exact for polynomials of degree ≤ 1. We obtain for $f(x) = 1$, , then $2 = c_1 + 3 \cdot 1 \Rightarrow c_1 = -1$.
For $f(x) = x$, then $2 = (-1) \cdot 0 + 3c_2 \Rightarrow c_2 = \frac{2}{3}$. Therefore $c_1 + 3c_2 = -1 + 3 \cdot \frac{2}{3} = 1$.

86. Let $p(x)$ be the polynomial of degree at most 2 that interpolates and data $(-1, 2)$, $(0, 1)$ and $(1, 2)$. If $q(x)$ is a polynomial of degree at most 3 such that $p(x) + q(x)$ interpolates and data $(-1, 2)$, $(0, 1)$, $(1, 2)$ and $(2, 11)$, then $q(3)$ equals——-? **GATE(MA): 2017**
Solution: By Lagrange interpolation, we have

$$\begin{aligned}
& p(x) = \frac{x^2 - x}{2} \cdot 2 + \frac{x^2 - 1}{-1} \cdot 1 + \frac{x^2 + x}{2} \cdot 2 = x^2 + 1 \\
\text{and} \quad & p(x) + q(x) = \frac{x(x-1)(x-2)}{(-1)(-2)(-3)} \cdot 2 + \frac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} \cdot 1 \\
+ \quad & \frac{(x+1)x(x-2)}{(2)(1)(-1)} \cdot 2 + \frac{(x+1)(x-1)x}{(3)(2)(1)} \cdot 11 \\
= \quad & x^3 + x^2 - x + 1 \\
\therefore \quad & q(x) = x^3 - x \\
\text{so,} \quad & q(3) = 3^3 - 3 = 24
\end{aligned}$$

87. Let J be the Jacobi iteration matrix of the linear system $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
Consider the following statements: **GATE(MA): 2017**
P: One of the eigenvalues of J lies in the interval $[2, 3]$.
Q: The jacobi iteration converges for the above system.
Which of the above statements hold TRUE?
(a) Both P and Q (b) Only P (c) Only Q (d) Neither P nor Q.
Ans. (b)
Hint. Iteration matrix of the given linear system

$$J = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ -4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -2 \\ 4 & -2 & 0 \end{bmatrix}$$

For the eigenvalues of J:

$$|J - \lambda I| = 0 \Rightarrow \begin{bmatrix} -\lambda & -2 & -1 \\ -2 & -\lambda & -2 \\ 4 & -2 & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^3 - 4\lambda - 12 = 0$$

Let $f(\lambda) = \lambda^3 - 4\lambda - 12$, then $f(2) \cdot f(3) < 0$. Hence one of the eigenvalues of J lies in the interval $[2, 3]$. So statement (P) is true.
Next by the theorem, a necessary and sufficient condition for convergence of an iteration method is that the eigenvalues of iteration matrix satisfy $\lambda_i(J) < 1$. So statement Q is not true.

88. For the composition of $\sqrt{x+1} - 1$ at $x = 1.2345678 \times 10^{-5}$ using a machine which keeps 8 significant digits, which of the following equivalent expressions would be best to use

(a) $\sqrt{x+1}-1$ (b) $\left(1-\frac{1}{\sqrt{x-1}}\right)\sqrt{x+1}$ (c) $\frac{x}{\sqrt{x+1}+1}$ (d) $\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\cdots$

Ans. (d) **NET(DEC.): 2011**

Hint. We have $\sqrt{x+1}-1=(1+x)^{\frac{1}{2}}-1=\left(1+\frac{x}{2}+\frac{1}{2}(\frac{1}{2}-1)\frac{x^2}{2!}+\cdots\right)=\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\cdots$.

89. Consider the interpolation data given below

$x:$	1	0.5	3
$y:$	3	-10	2

The interpolating polynomial corresponding to this data is given by

(a) $p(x)=-3(x-\frac{1}{2})(x-3)-8(x-1)(x-3)+\frac{2}{5}(x-1)(x-\frac{1}{2})$ **NET(DEC.): 2011**

(b) $q(x)=3+26(x-1)-\frac{53}{5}(x-1)(x-\frac{1}{2})$

(c) $r(x)=-\frac{53}{5}x^2+\frac{419}{10}x-\frac{283}{10}$

(d) $p(x)+q(x)+r(x)$

Ans. (a), (b) and (c)

90. Let $A=\begin{bmatrix}1.01 & 0.99\\ 0.99 & 1.01\end{bmatrix}$, $b=\begin{bmatrix}2\\2\end{bmatrix}$ and x_0 be the unique solution of the equation $Ax=b$, Let $\hat{x}_1$ and $\hat{x}_2$ be the two approximate solutions $\begin{bmatrix}1.01\\1.01\end{bmatrix}$ and $\begin{bmatrix}2\\0\end{bmatrix}$. Finally let $r_1=Ax_0-A\hat{x}_1=b-A\hat{x}_1$ and $r_2=Ax_0-A\hat{x}_2=b-A\hat{x}_2$ be the corresponding residues. Which of the following is / are correct ?

(a) $\hat{x}_1$ is a good approximation to x_0 and r_1 is small. **NET(DEC.): 2011**

(b) $\hat{x}_1$ is not approximation to x_0 and r_1 is small.

(c) $\hat{x}_1$ is not a good approximation to x_0 but r_2 is small.

(d) $\hat{x}_2$ is not a good approximation to x_0 but r_2 is small.

Ans. (a), (d)

91. Given that, an upper triangular matrix (UTM) is invertible if and only if all its diagonal elements are different from zero, consider the linear system $\begin{array}{r}2x_1+3x_2-x_3=5\\ 4x_1+4x_2-3x_3=3\\ -2x_1+3x_2-x_3=1\end{array}$. Then the above system

(a) can be transformed into an UTM but is not invertible because the diagonal entries of the UTM are non different from zero **NET(Jun): 2012**

(b) is invertible though can not be transformed into an UTM

(c) can be transformed into an UTM because above diagonal entries are all different from zero

(d) can be transformed into an UTM and the solution of the UTM is the solution of above system

Ans. (d)

92. Consider the function $f(x)=x^2-x-2=0$. Let $x=g(x)$, so that any fixed point of $g(x)$ is a solution of $f(x)=0$. Then **NET(Jun): 2012**

(a) $g(x)=x-\frac{x^2-x-2}{m}$, $m\in[-a,a]$ is a possible choice, where a is a positive constant.

(b) $g(x)=x^2-2$, $g(x)=1+\frac{2}{x}$ are possible choices

(c) $g(x)=x-\frac{x^2-x-2}{K}$, $K\neq 0$, $K\in\mathbb{R}$ is a possible choice

(d) $g(x)=x^2-3$, $g(x)=1+\frac{2}{x}$ are only possible choices

Ans. (b) and (c).

Hint. Given that $f(x)=x^2-x-2=0$. Let $g(x)=x$, so that any fixed point of $g(x)$ is a solution of above equation.

(a) $g(x) = x - \frac{x^2-x-2}{m}$ is not defined at $m = 0$ but $m \in [-a, a]$ where a is a positive constant. Hence (a) is not correct.
(b) We have $g(x) = x^2 - 2$. Hence the fixed point of $g(x) = x^2 - 2$ are the solution of the equation $g(x) = x$ as $x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0$.
Also $g(x) = 1 + \frac{2}{x}$ is again a solution of $f(x) = 0$ as $1 + \frac{2}{x} = x \Rightarrow x^2 - x - 2 = 0$.
(c) We have, $g(x) = x - \frac{x^2-x-2}{K}$, $k \neq 0$, $K \in \mathbb{R}$ and Hence the fixed point of $g(x) = x - \frac{x^2-x-2}{K}$ are solution of equation $g(x) = x$ as $x - \frac{x^2-x-2}{K} = x \Rightarrow x^2 - x - 2 = 0$.
(d) $g(x) = x^2 - 3$ is not a solution of $f(x) = 0$ as $g(x) = x \Rightarrow x^2 - x - 3 = 0$.

93. Let f be continuous map from the interval $[0, 1]$ into itself and consider the iteration $x_{n+1} = f(x_n)$. Which of the following maps will yield a fixed point for f? **NET(Dec.): 2012**
(a) $f(x) = \frac{x^2}{4}$ (b) $f(x) = \frac{x^2}{8}$ (c) $f(x) = \frac{x^2}{16}$ (d) $f(x) = \frac{x^2}{32}$
Ans. (a), (b), (c) and (d).

94. Consider the ordinary differential equation

$$\frac{dy}{dx} = \lambda y,\ t > 0,\ y(0) = 1$$

and the Euler scheme with step size h,

$$\frac{Y_{n+1} - Y_n}{h} = \lambda Y_n,\ n \geq 1,\ Y_0 = 1.$$ **NET(Dec.) : 2012**

Which of the following are necessarily true for Y_1 which approximates $Y(h) = e^{\lambda h}$?
(a) Y_1 is a polynomial approximation (b) Y_1 is a rational function approximation
(c) Y_1 is a trigonometric function approximation (b) Y_1 is a truncation of infinite series
Ans. (a) and (d).

95. Let $f(u) = u^3 - u - 1$. **NET(Dec.): 2012**
(a) Starting with the initial guess $u^{(0)} = 15$, the fixed point iterates of the equation $u = g(u)$, where $g(u) = u^3 - 1$ converge
(b) Starting with the initial guess $u^{(0)} = 15$, the fixed point iterates of the equation $u = \tilde{g}(u)$, where $\tilde{g}(u) = \sqrt{1 + u^3}$ converge
(c) If u^* is a root of the equation $f(u) = 0$ and $u^* > 1$ is a stable fixed point of the equation $u = g(u)$
(d) $f(u) = 0$ has a root between 1 and 2.
Ans. (d). (See MCQ 92)

96. To compute the value of e^t in the interval $[0, 1]$, pick $t_1 = 0$, $t_2 = 0.5$ and $t_3 = 1$. Let p be the quadratic polynomial that interpolates e^t, i.e., $p(t_i) = e^t$, $i = 1, 2, 3$. Then **NET(Dec.): 2012**
(a) the polynomial p can be written in the form $L_1(t) + e^{\frac{1}{2}}L_2(t) + eL_3(t)$ for some choice of quadratic polynomials L_1, L_2, L_3.
(b) If the polynomial p is written in the form $L_1(t) + e^{\frac{1}{2}}L_2(t) + eL_3(t)$ where L_1, L_2, L_3 are polynomials, then L_1, L_2, L_3 are uniquely determined.
(c) If p is written in the form $L_1(t) + e^{\frac{1}{2}}L_2(t) + eL_3(t)$ then one of L_1, L_2 or L_3 must be linear.
(d) the polynomial p is uniquely determined.
Ans. (a), (b) and (d).

97. Consider a sufficiently smooth function $f(x)$. A formula for estimating its derivative is given by $\frac{df}{dx} = \frac{1}{4h}\Big[f(x+2h) - f(x-2h)\Big] + \textit{error term}$, where $h > 0$. Let $f^{(n)}$ denote the n-th derivative of f and let ξ be a point between $x - 2h$ and $x + 2h$. Which of the following expressions for the error term are correct? **NET(Jun): 2013**
(a) $-\frac{f^2(\xi)h^2}{2}$ (b) $-\frac{2f^3(\xi)h^2}{3}$ (c) $-f^1(\xi)h$ (d) $-\frac{f^4(\xi)h^4}{12}$.
Ans. (b).

98. Consider the initial value problem

$$\frac{dy}{dx} = x + y, \;\; y(0) = 1.$$

Then the approximate value of the solution $y(x)$ at $x = 0.2$, using improved Euler method, with $h = 0.2$ is **NET(Dec): 2013**
(a) 1.11 (b) 1.20 (c) 1.24 (d) 1.48.
Ans. (c).

99. Let $y(x)$ satisfy the differential equation

$$\frac{dy}{dx} = \lambda y, \; t > 0, \; y(0) = 1$$

and then the backward Euler Method for $n \geq 1$ and $h > 0$.

$$\frac{y_n - y_{n-1}}{h} = \lambda y_n, \; n \geq 1, \; y_0 = 1. \qquad \textbf{NET(Dec.) : 2014}$$

yields
(a) a first order approximation to $e^{\lambda nh}$ (b) a polynomial approximation to $e^{\lambda nh}$
(c) a rational function approximation to $e^{\lambda nh}$ (b) a Chebyshev polynomial approximation to $e^{\lambda nh}$
Ans. (a) and (c).

100. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function. Which of the following is correct?
(a) There is $x_0 \in [0, \infty)$ such that $f(x_0) = x_0$ **NET(Dec.): 2015**
(b) If $f(x) \leq M$ for all $x \in [0, \infty)$ for some $M > 0$, then there exists $x_0 \in [0, \infty)$ such that $f(x_0) = x_0$.
(c) If f has a fixed point, then it must be unique
(d) f does not have a fixed point unless it is differentiable on $[0, \infty)$.
Ans. (b)

101. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with non-vanishing derivative. The Newton's method for finding a root of $f(x) = 0$ is the same as **NET(MS)(Dec)-2014**
(a) fixed pount iteration for the map $g(x) = x - \frac{f(x)}{f'(x)}$
(b) forward Euler method with unit step length for the differential equation $\frac{dy}{dt} + \frac{f(y)}{f'(y)} = 0$
(c) fixed point iteration for $g(x) = x + f(x)$
(d) fixed point iteration for $g(x) = x - f(x)$
Ans. (a) and (b).

102. Consider the iteration function for Newton's method $g(x) = x - \frac{f(x)}{f'(x)}$ and its application to find (approximate) square root of 2, starting with $x_0 = 2$. Consider the first and second iterates x_1 and x_2 respectively, then **NET(MS)(Jun)-2011**

((a) $1.5 < x_1 \le 2$ (b) $1.5 \le x_1 < 2$ (c) $x_1 \le 1.5$, $x_2 \le 1.5$ (d) $x_1 = 1.5$, $x_2 < 1$
Ans. (c).

103. Let $f(x) = ax + b$ for $a, b \in \mathbb{R}$. Then the iteration $x_{n+1} = f(x_n)$ starting from any given x_0 for $n \ge 0$ converges **NET(MS)(Jun)-2014**
(a) for all $a \in \mathbb{R}$ (b) for no $a \in \mathbb{R}$ (c) for $a \in [0, 1]$ (d) only for $a = 0$.
Ans. (c).

104. Let $f(x) = ax + 100$ for $a \in \mathbb{R}$. Then the iteration $x_{n+1} = f(x_n)$ for $n \ge 0$ and $x_0 = 0$ converges for **NET(MS)(Dec)-2015**
(a) $a = 5$ (b) $a = 1$ (c) $a = 0.1$ (d) $a = 10$.
Ans. (c).

3.3 Review Exercises

1 Solve by Euler's method the following differential equation

$$\frac{dy}{dx} = x + y + xy, y(0) = 1$$

and taking $h = 0.02$. calculate y at $x = 0.1$.
[**Ans.** 1.12] **CS-312/12**

2 Solve by Euler's method the following differential equation $\frac{dy}{dx} = xy, y(0) = 1$ and taking $h = 0.02$. calculate y at $x = 1$.
[**Ans.** 1.4593] **CS-312/08, 09**

3 Find the values of $y(0.10), y(0.15)$ using Euler's method Method taking $h = 0.05$, given that

$$\frac{dy}{dx} = x^2 + y^2, y(0) = 0$$

[**Ans.** 0.000125, 0.000625] **CS-312/07**

4 Compute $y(0.2)$

$$\frac{dy}{dx} = x + y, y(0) = 1$$

using Runge Kutta Method of 4th order correct up to three decimal places, taking step length $h = 0.1$.
[**Ans.** 1.2205] **WBUT-07**

5 Solve the initial value problem

$$10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$$

For $x = 0.1, 0.2$ by using Runge-Kutta Method of 4th order correct up to d decimal places.
[**Ans.** 1.0067, 1.0172] **WBUT-04**

6 Compute $y(0.2), y(0.4)$

$$\frac{dy}{dx} = 1 + y^2, y(0) = 0$$

using Runge Kutta Method of 4 th order correct up to three decimal places, taking step length $h = 0.1$.
[**Ans.** 0.851, 0.780] **CS-312/10**

7 Find the value of $y(0.1), y(0.2)$ and $y(0.3)$

$$\frac{dy}{dx} = xy + y^2, y(0) = 1$$

using Runge Kutta Method of 4th order correct up to three decimal places, taking step length $h = 0.1$.
[**Ans.** 1.1138, 1.2689, 1.4856] **CS-312/9**

8 Solve by Taylor's series method

$$\frac{dy}{dx} = \frac{1}{x^2 + y}, y(4) = 4$$

Compute the values of (4.1).
[**Ans.** 4.004] **M(CS)-312/11**

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