INTRODUCTION TO
DIFFERENTIAL EQUATIONS

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To my beloved Daughters

Samadrita & Somdatta

Preface

With the remarkable advancement in various branches of science, engineering and technology, today more than ever before, the study of differential equations has become essential. For, to have an exhaustive understanding of subjects like physics, mathematical biology, chemical science, mechanics, fluid dynamics, head transfer, aerodynamics, electricity, waves and electromagnetic, the knowledge of finding solution to differential equations is absolutely necessary. These differential equations may be ordinary or partial. Finding and interpreting their solutions are at the heart of applied mathematics. A thorough introduction to differential equations is therefore a necessary part of the education of any applied mathematician, and this book is aimed at building up skills in this area.

This book on ordinary / partial differential equations is the outcome of a series of lectures delivered by me, over several years, to the undergraduate or postgraduate students of Mathematics at various institution. My principal objective of the book is to present the material in such a way that would immediately make sense to a beginning student. In this respect, the book is written to acquaint the reader in a logical order with various well-known mathematical techniques in differential equations. Besides, interesting examples solving JAM / GATE / NET / IAS / SSC questions are provided in almost every chapter which strongly stimulate and help the students for their preparation of those examinations from graduate level.

Organization of the book
The book has been organized in a logical order and the topics are discussed in a systematic manner. It has comprising 21 chapters altogether. In the chapter **??**, the fundamental concept of differential equations including autonomous/ non-autonomous and linear / non-linear differential equations has been explained. The order and degree of the ordinary differential equations (ODEs) and partial differential equations(PDEs) are also mentioned. The chapter **??** are concerned the first order and first degree ODEs. It is also written in a progressive manner, with the aim of developing a deeper understanding of ordinary differential equations, including conditions for the existence and uniqueness of solutions. In chapter **??** the first order and higher degree ODEs are illustrated with sufficient examples. The chapter **??** is concerned with the higher order and first degree ODEs. Several methods, like method of undetermined coefficients, variation of parameters and Cauchy-Euler equations are also introduced in this chapter. In chapter **??**, second order initial value problems, boundary value problems and Eigenvalue problems with Sturm-Liouville problems are expressed with proper examples. Simultaneous linear differential equations are studied in chapter **??**. It is also written in a progressive manner with the aim of developing some alternative methods. In chapter **??**, the equilibria, stability

and phase plots of linear / nonlinear differential equations are also illustrated by including numerical solutions and graphs produced using Mathematica version 9 in a progressive manner. The geometric and physical application of ODEs are illustrated in chapter **??**. The chapter **??** is presented the Total (Pfaffian) Differential Equations. In chapter **??**, numerical solutions of differential equations are added with proper examples. Further, I discuss Fourier transform in chapter **??**, Laplace transformation in chapter 1, Inverse Laplace transformation in chapter 2. Moreover, series solution techniques of ODEs are presented with Frobenius method in chapter 3, Legendre function and Rodrigue formula in Chapter 4, Chebyshev functions in chapter **??**, Bessel functions in chapter 5 and more special functions for Hypergeometric, Hermite and Laguerre in chapter **??** in detail. Green function and application of ODE are developed in Chapters 20 and 21 respectively.

Besides, the partial differential equations are presented in chapter **??**. In the said chapter, the classification of linear, second order partial differential equations emphasizing the reasons why the canonical examples of elliptic, parabolic and hyperbolic equations, namely Laplace's equation, the diffusion equation and the wave equation have the properties that they do has been discussed. Also all chapters are concerned with sufficient examples. In addition, there is also a set of exercises at the end of each chapter to reinforce the skills of the students.

By reading this book, I hope that the readers will appreciate and be well prepared to use the wonderful subject of differential equations.

Aim and Scope
When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results of the modelling process is a system of ordinary or partial differential equations. Finding and interpreting the solutions of these differential equations is therefore a central part of applied mathematics, Physics and a thorough understanding of differential equations is essential for any applied mathematician and physicist. The aim of this book is to develop the required skills on the part of the reader. The book will thus appeal to undergraduates/postgraduates in Mathematics, but would also be of use to physicists and engineers. There are many worked examples based on interesting real-world problems. A large selection of examples / exercises including JAM/NET/GATE questions is provided to strongly stimulate and help the students for their preparation of those examinations from graduate level. The coverage is broad, ranging from basic ODE , PDE to second order ODE's including Bifurcation theory, Sturm-Liouville theory, Fourier Transformation, Laplace Transformation and existence and uniqueness theory, through to techniques for nonlinear differential equations including stability methods. Therefore, it may be used in research organization or scientific lab.

Significant features of the book

- A complete course of differential Equations
- Perfect for self-study and class room
- Useful for beginners as well as experts
- More than 500 worked out examples
- Large number of exercises
- More than 600 multiple choice questions with answers
- Suitable for GATE, NET, JAM, JEST, IAS, SSC examinations.

Use of software
The software package Latex version 5.3 was used to write the book. Mathematica version 9 was used to obtain the phase curve, eigenvalue for checking the stability of a dynamical system and solve the different equations. Lingo version 8 was also some time used to obtain the numerical results. All these packages were able to solve problems in material requirements planning and project management techniques easily.

ACKNOWLEDGEMENTS

This book is the outcome of a series of lectures and research experience carried out by me over several years. However it would not be possible to incorporate or framing the entire book without the help of many academicians. As such, I am indebted to many of my teachers and students. Especially I would like to thank Dr. Swapan Kumar Misra, Principal, Mugberia Gangadhar Mahavidyalaya for his generous support in writing this book.

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Without the unfailing love and support of my parents, who have always believed in me, this work would not have been possible. In addition, the care, love, patience, and understanding of my wife and lovely daughters have been of inestimable encouragement and help. I love them very much and appreciate all that they have contributed to my work.

In written this book, I have taken some helps from several books and research papers which are immensely helpful and are given in an alphabetical order in the bibliography. So I also express my gratitude and respect to all the eminent authors. I apologize if I inadvertently missed acknowledging mygratitude to any one else.

I shall feel great to receive constructive criticisms through email for the improvement of the book from the experts as well as the learners.

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Contents

Chapter 1

Laplace Transformation

1.1 Introduction

Laplace transform, one of the most important integral transforms is essentially a mathematical tool which can be used to solve several problems in engineering and science. The said transform was first introduced by Pierr Simon de Laplace (1749-1827), a French mathematician, in the year 1790 in his work on probability theory. This technique become popular when Heaviside applied to the solution of an ordinary differential equation referred hereafter as ordinary differential equation, representing a problem in electrical engineering. Furthermore, the method of Laplace transform is preferable and advantageous in solving linear ordinary differential equations with the right-hand side functions involving discontinuous and impulse functions. It also has applications in quantum mechanics, fractional calculus, etc.

In this chapter, we have presented the formal definition of the Laplace transform and calculate the Laplace transforms of some elementary functions directly from the definition. The basic operational properties of the Laplace transforms including convolution and its properties and the differentiation and integration of Laplace transforms are discussed in some detail. The Heaviside Expansion Theorem for the Laplace transform are discussed.

1.2 Definition of Laplace transformation

Definition 1.1 Let $f(t)$ be defined for $0 \leq t < \infty$ and let s denoted an arbitrary real variable. The Laplace transform of $f(t)$, designated by either $L\Big\{f(t)\Big\}$ or $F(s)$, is

$$L\Big\{f(t)\Big\} = F(s) = \int_0^\infty e^{-st} f(t)dt, \ \ \forall s > 0, \tag{1.1}$$

such that the improper integral converges. Convergence occurs when the limit exists i.e.,

$$\lim_{X\to\infty} \int_0^X e^{-st} f(t)dt \text{ exists.} \tag{1.2}$$

If the limit does not exist, the improper integral diverges and $f(t)$ has no Laplace transform. When evaluating the integral in equation 1.1, the variable s is treated as a constant because the integration is with respect to t.

Comment: Laplace transform is defined for complex value function $f(t)$ and the parameter s can also be complex. But we restrict our discussion only for case in which $f(t)$ is real value and s is real.

1.3 Existence of the Laplace Transform

Theorem 1.1 Existence Theorem
Let us consider a functions $f(t)$ which possess finite discontinuities, because in the applications of Laplace transforms to physical problems these frequently arise. Examples are the unit Heaviside step function and the unit rectangular wave function illustrated in Fig.-1.1. Such functions are said to be piecewise continuous.

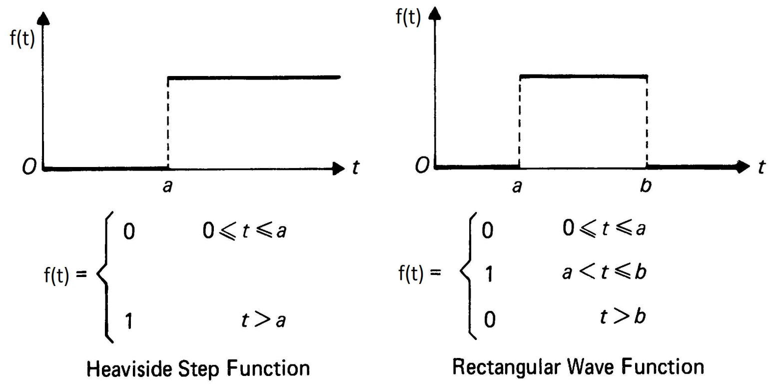

Figure 1.1:

Definition 1.2 Piecewise continuity; A function f of t is piecewise continuous in the closed interval $a \leq t \leq b$ when the interval can be subdivided into a finite number of subintervals, $a \leq t \leq t_1, t_1 \leq t \leq t_2, \cdots, t_{n-1} \leq t \leq b$ such that
(i) $f(t)$ is continuous in each open interval

$$t_r < t < t_{r+l}, \qquad r = 0, 1, \cdots, n-1, t_0 = a, t_n = b.$$

(ii) $f(t)$ tends to a finite limit as t tends to each end point from within the interval, i.e. for small $\epsilon > 0$

$$\lim_{\epsilon \to 0} F(t_r + B) = F(t_r+), \lim_{\epsilon \to 0} f(t_{r+1} - \epsilon) = F(t_{r+1}-)$$

exist for all $r = 0, 1, \cdots, n-1$.

A function which is piecewise continuous in a finite interval is integrable over that interval. The next step is to obtain sufficient conditions on $f(t)$ in order that $L\{f(t)\}$ exists. Since $\int_T^\infty e^{-(s-\sigma)t} dt, (T \geq 0)$ converges for $s > \sigma$, we can use this fact to explore the convergence of the Laplace transform integral $\int_0^\infty e^{-st} f(t) dt$ when $|e^{-\sigma t} f(t)| < M$ for all $t \geq T$, T and M being positive constants.

Definition 1.3 Exponential Order A function f of t is of exponential order σ as $t \to \infty$ if constants $\sigma, M(> 0)$ and $T(> 0)$ can be found such that

$$|e^{-\sigma t} f(t)| < M \text{ or } |f(t)| < Me^{\sigma t}, \text{ for all } t \geq T > 0$$

Equivalently, we write

$$F(t) = O(e^{\sigma t}) \text{ as } t \to 0.$$

Existence of the Laplace Transform

A function f has a Laplace transform whenever it is of **exponential order**. That is, there must be a real number B such that

$$\lim_{t\to\infty} \left| f(t)e^{-Bt} \right| = 0 \tag{1.3}$$

As an example, every exponential function $Ae^{\alpha t}$ has a Laplace transform for all finite values of A and α. The canonical form of an exponential function, as typically used in signal processing, is

$$a(t) = Ae^{-t/\tau}, \quad t \geq 0$$

where τ is called the time constant of the exponential. A is the peak amplitude. The time constant is the time it takes to decay by $1/e$, i.e.,

$$\frac{a(\tau)}{a(0)} = \frac{1}{e}.$$

Sufficient Condition for existence of Laplace transform:

Let f be a piecewise continuous function in $[0, \alpha)$ and is of exponential order. Then Laplace transform $F(s)$ of f exists for $s > c$, where c is a real number that depends on f.

Proof. Since f is of exponential order, there exist a, M, σ such that $|f(t)| \leq Me^{\sigma t}$ for $t \geq A$. Now we write $I = \int_0^\infty f(t)e^{-st}dt = I_1 + I_2$ where $I_1 = \int_0^A f(t)e^{-st}dt$ and $I_2 = \int_A^\infty f(t)e^{-st}dt$. Since f is piecewise continuous, I_1 exists. For the second integral I_2, we note that for $t \geq A$, $|e^{-st}f(t)| \leq Me^{-(s-\sigma)t}$. Thus $\int_A^\infty |f(t)e^{-st}|dt \leq \int_A^\infty e^{-(s-\sigma)t}dt \leq \int_0^\infty e^{-(s-\sigma)t}dt = \frac{M}{s-\sigma}$, $s > \sigma$. Since the integral in I_2 converges absolutely for $s > \sigma$, I_2 converges for $s > \sigma$. Thus both I_1 and I_2 exist and hence I exists for $s > \sigma$.
Comment The above condition is not necessary. For example consider $f(t) = \frac{1}{\sqrt{t}}$ which is not piecewise continuous in $[0, \infty)$. But $\int_0^\infty \frac{e^{-st}}{\sqrt{t}} = \sqrt{\frac{\pi}{s}}$, $s > 0$.

Example 1.1 If $f(t) = 1$ for $t > 0$ and then $L\{1\} = \frac{1}{s}, s > 0$.

Solution: From the definition of Laplace Transform, we have

$$\begin{aligned} L\Big\{f(t)\Big\} &= \int_0^\infty e^{-st} f(t)dt \\ \Rightarrow L\Big\{1\Big\} &= \lim_{X\to\infty} \int_0^X e^{-st} \cdot 1dt = \lim_{X\to\infty} \Big[-\frac{e^{-sX}}{s} + \frac{1}{s} \Big] = \frac{1}{s}, s > 0 \end{aligned}$$

Example 1.2 If $f(t) = e^{at}$ for $t > 0$ and then $L\left\{e^{at}\right\} = \frac{1}{s-a}, s > a$.

Solution: From the definition of Laplace Transform, we have

$$\begin{aligned} L\left\{f(t)\right\} &= \int_0^\infty e^{-st} f(t)\, dt \\ \Rightarrow L\left\{e^{at}\right\} &= \lim_{X\to\infty} \int_0^X e^{-st} \cdot e^{at}\, dt \\ &= \lim_{X\to\infty} \int_0^X e^{-(s-a)t} dt = \lim_{X\to\infty} \left[-\frac{e^{-(s-a)X}}{s-a} + \frac{1}{s-a} \right] = \frac{1}{s-a}, \ \because s > a \end{aligned}$$

Example 1.3 If $f(t) = t$ for $t > 0$ and then $L\left\{t\right\} = \frac{1}{s^2}, s > 0$.

Solution: From the definition of Laplace Transform, we have

$$\begin{aligned} L\left\{f(t)\right\} &= \int_0^\infty e^{-st} f(t) dt \\ \Rightarrow L\left\{t\right\} &= \lim_{X\to\infty} \int_0^X e^{-st} \cdot t\, dt = \lim_{X\to\infty} \left[-X\frac{e^{-sX}}{s} - \frac{e^{-sX}}{s^2} + \frac{1}{s^2} \right] = \frac{1}{s^2}, s > 0 \end{aligned}$$

Example 1.4 If $f(t) = t^n$ for $t > 0$, then $L\left\{t^n\right\} = \frac{\Gamma(n+1)}{s^{n+1}}, n > -1, s > 0$.

Solution: From the definition of Laplace Transform, we have

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\ \Rightarrow L\left\{t^n\right\} &= \lim_{X\to\infty} \int_0^X e^{-st} \cdot t^n\, dt \\ &= \lim_{X\to\infty} \int_0^{sX} \frac{z^n}{s^n}\, e^{-z}\, \frac{dz}{s} \left[\text{putting } st = z, \text{ then } dt = \frac{dz}{s} \right] \\ &= \frac{1}{s^{n+1}} \int_0^\infty z^{(n+1)-1}\, e^{-z}\, dz \\ &= \frac{\Gamma(n+1)}{s^{n+1}}, s > 0, n > -1, \left[\text{Since } \Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx \right] \end{aligned}$$

Example 1.5 If $f(t) = \sin at$, where a is a real constant, then, $L\left\{\sin at\right\} = \frac{a}{s^2+a^2}, s > 0$.

Solution: From the definition of Laplace Transform, we have

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt \\
\Rightarrow L\{\sin at\} &= \lim_{X\to\infty} \int_0^X e^{-st} \sin at\, dt = \lim_{X\to\infty} \int_0^X e^{-st} \frac{e^{iat} - e^{-iat}}{2i}\, dt \\
&= \frac{1}{2i} \lim_{X\to\infty} \int_0^X [e^{-t(s-ia)} - e^{-t(s+ia)}]\, dt \\
&= \frac{1}{2i} \lim_{X\to\infty} \left[-\frac{e^{-X(s-ia)}}{(s-ia)} + \frac{1}{s-ia} + \frac{e^{-X(s+ia)}}{(s+ia)} - \frac{1}{s+ia} \right] \\
&= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{a}{s^2+a^2},\ s > 0.
\end{aligned}$$

Example 1.6 If $f(t) = \cos at$, where a is a real constant, then,

$$L\Big\{ \cos at \Big\} = \frac{s}{s^2+a^2},\ s > 0.$$

Solution: From the definition of Laplace Transform, we have

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt \\
\Rightarrow L\{\cos at\} &= \lim_{X\to\infty} \int_0^X e^{-st} \cos at\, dt = \lim_{X\to\infty} \int_0^X e^{-st} \frac{e^{iat} + e^{-iat}}{2}\, dt \\
&= \frac{1}{2} \lim_{X\to\infty} \int_0^X [e^{-t(s-ia)} + e^{-t(s+ia)}]\, dt \\
&= \frac{1}{2} \lim_{X\to\infty} \left[-\frac{e^{-X(s-ia)}}{(s-ia)} + \frac{1}{s-ia} - \frac{e^{-X(s+ia)}}{(s+ia)} + \frac{1}{s+ia} \right] \\
&= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{s}{s^2+a^2},\ s > 0
\end{aligned}$$

Example 1.7 If $f(t) = \sinh at$, where a is a real constant, then, $L\Big\{ \sinh at \Big\} = \frac{a}{s^2-a^2},\ s > |a|$.

Solution: From the definition of Laplace Transform, we have

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt \\
\Rightarrow L\{\sinh at\} &= \lim_{X\to\infty} \int_0^X e^{-st} \sinh at\, dt = \lim_{X\to\infty} \int_0^X e^{-st} \frac{e^{at} - e^{-at}}{2}\, dt \\
&= \frac{1}{2} \lim_{X\to\infty} \int_0^X [e^{-t(s-a)} - e^{-t(s+a)}]\, dt \\
&= \frac{1}{2} \lim_{X\to\infty} \left[-\frac{e^{-X(s-a)}}{(s-a)} + \frac{1}{s-a} + \frac{e^{-X(s+a)}}{(s+a)} - \frac{1}{s+a} \right] \\
&= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2},\ s > |a|
\end{aligned}$$

Example 1.8 If $f(t) = \cosh at$, where a is a real constant, then, $L\Big\{\cosh at\Big\} = \frac{s}{s^2-a^2}, s > |a|$.

Solution: From the definition of Laplace Transform, we have

$$\begin{aligned}
L\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \\
\Rightarrow L\{\cosh at\} &= \lim_{X\to\infty} \int_0^{X} e^{-st} \cosh at \, dt \; = \lim_{X\to\infty} \int_0^{X} e^{-st} \frac{e^{at} + e^{-at}}{2} dt \\
&= \frac{1}{2} \lim_{X\to\infty} \int_0^{X} [e^{-t(s-a)} + e^{-t(s+a)}] \, dt \\
&= \frac{1}{2} \lim_{X\to\infty} \Big[-\frac{e^{-X(s-a)}}{(s-a)} + \frac{1}{s-a} - \frac{e^{-X(s+a)}}{(s+a)} + \frac{1}{s+a} \Big] \\
&= \frac{1}{2} \Big[\frac{1}{s-a} + \frac{1}{s+a} \Big] = \frac{s}{s^2 - a^2}, s > |a|.
\end{aligned}$$

1.4 Basic Properties of Laplace transformation

Theorem 1.2 Uniqueness of Laplace transform Let $f(t)$ and $g(t)$ be two functions such that $F(s) = G(s)$ for all $s > k$. Then $f(t) = g(t)$ at all t where both are continuous.

Property 1.1 Linearity Property:

$$L\Big\{c_1 f_1(t) + c_2 f_2(t)\Big\} = c_1 L\Big\{f_1(t)\Big\} + c_2 L\Big\{f_2(t)\Big\} = c_1 F_1(s) + c_2 F_2(s).$$

Proof: By the definition of Laplace Transformation

$$\begin{aligned}
L\Big\{c_1 f_1(t) + c_2 f_2(t)\Big\} &= \int_0^{\infty} \Big\{c_1 f_1(t) + c_2 f_2(t)\Big\} e^{-st} dt \\
&= \int_0^{\infty} \Big\{c_1 f_1(t)\Big\} e^{-st} dt + \int_0^{\infty} \Big\{c_2 f_2(t)\Big\} e^{-st} dt \\
&= c_1 \int_0^{\infty} \Big\{f_1(t)\Big\} e^{-st} dt + c_2 \int_0^{\infty} \Big\{f_2(t)\Big\} e^{-st} dt \\
&= c_1 L\Big\{f_1(t)\Big\} + c_2 L\Big\{f_2(t)\Big\} \\
&= c_1 F_1(s) + c_2 F_2(s) \qquad \textbf{(Hence proved)}
\end{aligned}$$

Property 1.2 Change of scale property: If $L\Big\{f(t)\Big\} = F(s)$, then

$$L\Big\{f(at)\Big\} = \frac{1}{a} F\Big(\frac{s}{a}\Big)$$

Proof: By the definition of Laplace Transformation

$$
\begin{aligned}
L\Big\{f(t)\Big\} &= \int_0^\infty f(t)e^{-st}dt \\
\Rightarrow L\Big\{f(at)\Big\} &= \lim_{X\to\infty}\int_0^X f(at)e^{-st}dt \qquad \text{put } at = y, dt = \frac{dy}{a} \\
&= \lim_{X\to\infty}\int_0^X f(y)e^{-\frac{s}{a}y}\frac{dy}{a} \\
&= \frac{1}{a}\int_0^\infty f(y)e^{-\frac{s}{a}y}dy \qquad = \frac{1}{a}F\Big(\frac{s}{a}\Big) \qquad \textbf{(Hence proved)}
\end{aligned}
$$

Property 1.3 First shifting property: If $L\Big\{f(t)\Big\} = F(s)$, then

$$L\Big\{e^{at}f(t)\Big\} = F(s-a) \text{ for } s > a$$

Proof: By the definition of Laplace Transformation

$$L\Big\{f(t)\Big\} = \int_0^\infty f(t)e^{-st}dt \Rightarrow L\Big\{e^{at}f(t)\Big\} = \lim_{X\to\infty}\int_0^X \Big\{e^{at}f(t)\Big\}e^{-st}dt$$

$$= \lim_{X\to\infty}\int_0^X f(t)e^{-(s-a)t}dt = \int_0^\infty f(t)e^{-(s-a)t}dt, s-a > 0$$

$$= F(s-a) \qquad \textbf{(Hence proved)}$$

1.5 Operational Rules of Laplace Transforms

Theorem 1.3 Division rule:
If $L\Big\{f(t)\Big\} = F(s)$, then $L\Big\{\frac{f(t)}{t}\Big\} = \int_s^\infty F(s)ds$, provided $\lim\limits_{t\longrightarrow\infty}\frac{f(t)}{t}$ exists finitely.

Proof: By the definition of Laplace Transformation

$$F(s) = L\Big\{f(t)\Big\} = \int_0^\infty f(t)e^{-st}dt \Rightarrow \int_s^\infty F(s)ds = \int_s^\infty \Big\{\int_0^\infty f(t)e^{-st}dt\Big\}ds$$

Interchanging the order of integration, we get

$$\int_s^\infty F(s)ds = \int_0^\infty \Big\{\int_s^\infty f(t)e^{-st}ds\Big\}dt = \int_0^\infty f(t)\Big\{\int_s^\infty e^{-st}ds\Big\}dt$$

$$= \int_0^\infty f(t)\Big\{\lim_{X\to\infty}\int_s^X e^{-st}ds\Big\}dt = \int_0^\infty f(t)\Big\{\lim_{X\to\infty}\Big[\frac{e^{-st}}{-t}\Big]_s^X\Big\}dt$$

$$= \int_0^\infty \frac{f(t)}{t}e^{-st}dt = L\Big\{\frac{f(t)}{t}\Big\}$$

$$\Rightarrow L\Big\{\frac{f(t)}{t}\Big\} = \int_s^\infty F(s)ds \qquad \textbf{(Hence proved)}$$

Theorem 1.4 Multiplication rule for t^n:
If $L\Big\{f(t)\Big\} = F(s)$, then $L\Big\{t^n f(t)\Big\} = (-1)^n \frac{d^n}{ds^n}\Big(F(s)\Big)$, where n is positive integer

Proof: By the definition of Laplace Transformation

$$
\begin{aligned}
&F(s) = L\Big\{f(t)\Big\} = \int_0^\infty f(t)e^{-st}dt \\
&\Rightarrow \frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt \;=\; \int_0^\infty f(t)(-t)e^{-st}dt \\
&= -\int_0^\infty \Big\{tf(t)\Big\}e^{-st}dt \;=\; -L\Big\{tf(t)\Big\} \\
&\Rightarrow L\Big\{tf(t)\Big\} = -\frac{dF(s)}{ds} \\
&\text{Similarly, } \; L\Big\{t^2 f(t)\Big\} = L\Big\{t\cdot tf(t)\Big\} \;=\; -\frac{d}{ds}L\Big\{tf(t)\Big\} \;=\; (-1)^2\frac{d}{ds}\left\{\frac{dF}{ds}\right\} \;=\; (-1)^2\frac{d^2F(s)}{ds^2}
\end{aligned}
$$

Hence the theorem is true for $n = 1, 2$. Let the theorem is true for $n = m$. Then

$$
\begin{aligned}
&L\Big\{t^m f(t)\Big\} = (-1)^m \frac{d^m}{ds^m}\Big(F(s)\Big) \\
&\Rightarrow \int_0^\infty \Big\{t^m f(t)\Big\}e^{-st}dt = (-1)^m \frac{d^m}{ds^m}\Big(F(s)\Big) \\
&\Rightarrow \frac{d}{ds}\int_0^\infty \Big\{t^m f(t)\Big\}e^{-st}dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\Big(F(s)\Big) \\
&\Rightarrow \int_0^\infty (-t)\Big\{t^m f(t)\Big\}e^{-st}dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\Big(F(s)\Big) \\
&\Rightarrow -\int_0^\infty \Big\{t^{m+1} f(t)\Big\}e^{-st}dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\Big(F(s)\Big) \\
&\Rightarrow L\Big\{t^{m+1} f(t)\Big\} = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\Big(F(s)\Big)
\end{aligned}
$$

This shows that the theorem is true for $n = m + 1$. Hence by Mathematical induction, we can say that the theorem is true for all integer n.

Theorem 1.5 Laplace transform of first order derivative:
If $L\Big\{f(t); s\Big\} = F(s)$, then $L\Big\{f'(t); s\Big\} = sF(s) - f(0)$, provided $f(t)$ is exponential order.

Proof: By the definition of Laplace Transformation

$$
\begin{aligned}
L\Big\{f(t);s\Big\} &= \int_0^\infty f(t)e^{-st}dt \\
\Rightarrow L\Big\{f'(t);s\Big\} &= \int_0^\infty f'(t)e^{-st}dt = \lim_{X\to\infty}\int_0^X f'(t)e^{-st}dt \\
&= \lim_{X\to\infty}\left\{\Big[e^{-st}f(t)\Big]_0^X + s\int_0^X f(t)e^{-st}dt\right\} \\
&= \lim_{X\to\infty}\left\{\Big[e^{-sX}f(X)-f(0)\Big] + s\int_0^X f(t)e^{-st}dt\right\} \\
&= 0 - f(0) + sF(s) \qquad \Big[\because \lim_{X\to\infty} e^{-sX}f(X) = 0\Big] \\
&= sF(s) - f(0)
\end{aligned}
$$

Theorem 1.6 Laplace transform of n-th order derivative:
If $L\Big\{f(t);s\Big\} = F(s)$, then

$$
\begin{aligned}
L\Big\{f''(t);s\Big\} &= s^2F(s) - sf(0) - f'(0) \\
L\Big\{f'''(t);s\Big\} &= s^3F(s) - s^2f(0) - sf'(0) - f''(0) \\
&\cdots \\
L\Big\{f^n(t);s\Big\} &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)
\end{aligned}
$$

provided $f(t)$ is exponential order.

Proof. Since

$$
\begin{aligned}
L\Big\{f'(t);s\Big\} &= sF(s) - f(0) \\
L\Big\{f''(t);s\Big\} &= s\,L\Big\{f'(t);s\Big\} - f'(0) = s(sF(s) - f(0)) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0) \\
L\Big\{f'''(t);s\Big\} &= s\,L\Big\{f''(t);s\Big\} - f''(0) = s(s^2F(s) - sf(0) - f'(0)) - f''(0) \\
&= s^3F(s) - s^2f(0) - sf'(0)) - f''(0)
\end{aligned}
$$

Thus in general,

$$
L\Big\{f^n(t);s\Big\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)
$$

Theorem 1.7 Laplace transform of partial derivative:
If u is a function of two variable $x,\ t$ and $U(x,s) = L[u(x,t);s]$, then prove that

$$\begin{aligned}
(i)\, L\left\{\frac{\partial u}{\partial t};s\right\} &= sU(x,s) - u(x,0)\\
(ii)\, L\left\{\frac{\partial^2 u}{\partial t^2};s\right\} &= s^2U(x,s) - s\,u(x,0) - u_t(x,0)\\
(iii)\, L\left\{\frac{\partial u}{\partial x};s\right\} &= \frac{dU(x,s)}{dx}\\
(iv)\, L\left\{\frac{\partial^2 u}{\partial x^2};s\right\} &= \frac{d^2U(x,s)}{dx^2}\\
(v)\, L\left\{\frac{\partial^2 u}{\partial x\partial t};s\right\} &= s\frac{dU(x,s)}{dx} - \frac{du(x,0)}{dx} \text{ provided } f(t) \text{ is exponential order.}
\end{aligned}$$

Proof: By the definition of Laplace Transformation, we have

$$\begin{aligned}
(i) L\left\{\frac{\partial u}{\partial t};s\right\} &= \int_0^\infty \frac{\partial u}{\partial t}e^{-st}dt = \lim_{X\to\infty}\int_0^X \frac{\partial u}{\partial t}e^{-st}dt\\
&= \lim_{X\to\infty}\left\{\Big[e^{-st}u(x,t)\Big]_0^X + s\int_0^X u(x,t)e^{-st}dt\right\}\\
&= -u(x,0) + s\int_0^\infty u(x,t)e^{-st}dt\\
\text{Therefore, } L\left\{\frac{\partial u}{\partial t};s\right\} &= sU(x,s) - u(x,0)\\
(ii) L\left\{\frac{\partial^2 u}{\partial t^2};s\right\} &= L\left\{\frac{\partial v}{\partial t};s\right\},\ v = \frac{\partial u}{\partial t}\\
&= s\Big\{L(v;s)\Big\} - v(x,0)\ = s\Big\{sU(x,s) - u(x,0)\Big\} - u_t(x,0)\\
\text{Therefore, } L\left\{\frac{\partial^2 u}{\partial t^2};s\right\} &= s^2U(x,s) - s\,u(x,0) - u_t(x,0)\\
(iii) L\left\{\frac{\partial u}{\partial x};s\right\} &= \int_0^\infty \frac{\partial u}{\partial x}e^{-st}dt\\
&= \frac{d}{dx}\int_0^\infty e^{-st}u(x,t)\,dt\ = \frac{dU(x,s)}{dx}\\
\text{Therefore, } L\left\{\frac{\partial u}{\partial x};s\right\} &= \frac{dU(x,s)}{dx}\\
(iv) L\left\{\frac{\partial^2 u}{\partial x^2};s\right\} &= \int_0^\infty \frac{\partial^2 u}{\partial x^2}e^{-st}dt\\
&= \frac{d^2}{dx^2}\int_0^\infty e^{-st}u(x,t)\,dt\ = \frac{d^2U(x,s)}{dx^2}\\
\text{Therefore, } L\left\{\frac{\partial^2 u}{\partial x^2};s\right\} &= \frac{d^2U(x,s)}{dx^2}
\end{aligned}$$

$$
\begin{aligned}
(v)\, L\left\{\frac{\partial^2 u}{\partial x\, \partial t}; s\right\} &= \int_0^\infty \frac{\partial^2 u}{\partial x\, \partial t} e^{-st} dt \\
&= \frac{d}{dx} \int_0^\infty \frac{\partial u}{\partial t} e^{-st}\, dt \\
&= \frac{d}{dx}\Big[sU(x,s) - u(x,0) \Big] \\
&= s\frac{dU(x,s)}{dx} - \frac{du(x,0)}{dx} \\
\text{Therefore, } L\left\{\frac{\partial^2 u}{\partial x \partial t}; s\right\} &= s\frac{dU(x,s)}{dx} - \frac{du(x,0)}{dx}
\end{aligned}
$$

Thus, we show that from the above results that the partial derivatives are transformed into ordinary derivatives.

Theorem 1.8 Laplace Transform of Integrals: If $L\Big\{f(t)\Big\} = F(s)$, then

$$L\left\{\int_0^t f(\xi)d\xi\right\} = \frac{F(s)}{s},$$

provided $f(t)$ is exponential order.

Proof: Let $G(t) = \int_0^t f(\xi)d\xi$. Therefore $G'(t) = f(t)$ and $G(0) = 0$. Taking Laplace Transformation in both sides of $G'(t) = f(t)$, we get

$$L\Big\{G'(t)\Big\} = L\Big\{f(t)\Big\} \Rightarrow sG(s) - G(0) = F(s) \qquad [\because G(0) = 0]$$

$$\Rightarrow G(s) = \frac{F(s)}{s} \;\Rightarrow L\left\{\int_0^t f(\xi)d\xi\right\} = \frac{F(s)}{s}$$

Theorem 1.9 Laplace transform of Periodic function:
If f be a periodic function with period $T(> 0)$, then

$$L\Big\{f(t)\Big\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)dt.$$

Proof: By the definition of Laplace Transformation

$$\begin{aligned}
L\Big\{f(t)\Big\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt + \cdots \\
&= \sum_{i=1}^{\infty}\int_{(i-1)T}^{iT} f(t)e^{-st}dt, \qquad \text{putting } t = u + (i-1)T, dt = du \\
&= \sum_{i=1}^{\infty}\int_0^T f(u+(i-1)T)e^{-s(u+(i-1)T)}du \\
&= \sum_{i=1}^{\infty}\int_0^T f(u+(i-1)T)e^{-s(u+(i-1)T)}du \qquad \because f(u+(i-1)T) = f(u) \\
&= \sum_{i=1}^{\infty} e^{-s(i-1)T)}\int_0^T f(u)e^{-su}du \qquad \because \sum_{i=1}^{\infty} e^{-s(i-1)T)} = \frac{1}{1-e^{-sT}} \\
&= \frac{1}{1-e^{-sT}}\int_0^T f(u)e^{-su}du
\end{aligned}$$

Example 1.9 Given that

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}$$

and extended periodically with period 2π. Find $L\Big\{f(t)\Big\}$?.

Solution: We know that

$$\begin{aligned}
L\Big\{f(t)\Big\} &= \frac{1}{1-e^{-sT}}\int_0^T f(u)e^{-st}dt = \frac{1}{1-e^{-2\pi s}}\Big\{\int_0^\pi \Big(\sin t\Big)e^{-st}dt + \int_\pi^{2\pi}\Big(0\Big)e^{-st}dt\Big\} \\
&= \frac{1}{1-e^{-2\pi s}}\Big\{\int_0^\pi \Big(\sin t\Big)e^{-st}dt\Big\} = \frac{1}{1-e^{-2\pi s}}\Big[\frac{e^{-st}}{s^2+1}\Big\{-s\sin t - \cos t\Big\}\Big]_0^\pi \\
&= \frac{1}{1-e^{-2\pi s}} \times \frac{e^{-\pi s}+1}{s^2+1} = \frac{1}{(1-e^{-\pi s})(s^2+1)}
\end{aligned}$$

Theorem 1.10 Convolution theorem: If $f * g = \int_0^u f(u-t)g(t)dt$, then

$$L\Big\{f * g\Big\} = F(s)\cdot G(s).$$

Proof. From the definition of the Laplace transform, we know that

$$\begin{aligned}
F(s)\,G(s) &= [\int_0^\infty f(t)e^{-st}dt]\,[\int_0^\infty g(v)e^{-sv}dv] \\
&= \int_0^\infty\int_0^\infty e^{-s(t+v)}f(t)g(v)dt\,dv \\
&= \int_0^\infty g(t)\{\int_0^\infty e^{-s(t+v)}f(v)\,dv\}\,dt
\end{aligned}$$

Let $t + v = u$ in the inner integral. Then

$$\begin{aligned} F(s)\,G(s) &= \int_0^\infty g(t)\{\int_t^\infty e^{-su} f(u-t)\,du\}\,dt \;=\; \int_0^\infty \{\int_0^u e^{-su} f(u-t) g(t)\,dt\}\,du \\ &= \int_0^\infty e^{-su}\{\int_0^u f(u-t) g(t)\,dt\}\,du \;=\; L[\int_0^u f(u-t) g(t)\,dt; u] \end{aligned}$$

Theorem 1.11 Initial and final value theorem: If $f(t)$ and $f'(t)$ are Laplace transformable and $F(s)$ is the Laplace transform of $f(t)$, then the behavior of $f(t)$ in the neighborhood of $t = 0$ corresponds to the behavior of $sF(s)$ in the neighborhood of $s = \infty$. Mathematically,

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s).$$

$$\begin{aligned} \textbf{Proof. } \text{Since } L\Big\{f'(t); s\Big\} &= sL\Big\{f(t); s\Big\} - f(0) \\ \text{So } \int_0^\infty f'(t)\,e^{-st} dt &= sF(s) - f(0) \end{aligned}$$

Taking the limit as $s \to \infty$ on both sides of the above equation, we have

$$\lim_{s\to\infty} \int_0^\infty e^{-st} f'(t) dt = \lim_{s\to\infty} sF(s) - \lim_{s\to\infty} f(0) \tag{1.4}$$

Since s is independent of t, so we take the limit before integrating the left-hand side of equation (1.4), thus getting

$$\lim_{s\to\infty} \int_0^\infty e^{-st} f'(t) dt = \int_0^\infty [\lim_{s\to\infty} e^{-st} f'(t)] dt = 0 \tag{1.5}$$

and using equation (1.5), the equation (1.4) becomes

$$\lim_{t\to\infty} s\,F(s) = f(0) = \lim_{t\to 0} f(t). \text{ Hence the theorem.}$$

Example 1.10 Verify initial value theorem for the function f defined by $f(t) = 1 + e^{-t}$.

Solution: Given $f(t) = 1 + e^{-t}$, we have

$$\begin{aligned} F(s) = L\Big\{f(t); s\Big\} &= L\Big\{1; s\Big\} + L\Big\{e^{-t}; s\Big\} = \frac{1}{s} + \frac{1}{s+1} \\ \text{So, } s\,F(s) &= 1 + \frac{s}{s+1} \end{aligned}$$

$$\therefore \lim_{s\to\infty} s\,F(s) = 2 = f(0) = \lim_{t\to 0} f(t). \text{ Hence the result.}$$

Theorem 1.12 The Final value theorem states:
If $f(t)$ and $f'(t)$ are Laplace transformable and $F(s)$ is the Laplace transform of $f(t)$, then the behavior of $f(t)$ in the neighborhood of $t = \infty$ corresponds to the behavior of $sF(s)$ in the neighborhood of $s = 0$. Mathematically,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s).$$

Proof. Since
$$\begin{aligned} L\Big\{f'(t); s\Big\} &= sL\Big\{f(t); s\Big\} - f(0) \\ \text{So } \int_0^\infty f'(t)\,e^{-st}dt &= sF(s) - f(0) \end{aligned}$$

Taking the limit as $s \to 0$ on both sides of the above equation, we have

$$\lim_{s\to 0}\int_0^\infty e^{-st} f'(t)dt = \lim_{s\to 0} sF(s) - \lim_{s\to 0} f(0) \tag{1.6}$$

$$\text{But } \int_0^\infty \lim_{s\to 0} e^{-st} f'(t)dt = \int_0^\infty f'(t)dt = \lim_{t\to\infty} f(t) - f(0) \tag{1.7}$$

Using the equations (1.6)-(1.7), we get' $\lim_{s\to 0} s\,F(s) = \lim_{t\to\infty} f(t)$. (1.8)

Example 1.11 Verify final value theorem for the function f defined by $f(t) = e^{-t}$.

Solution: Given $f(t) = e^{-t}$, we have

$$F(s) = L\Big\{f(t); s\Big\} = L\Big\{e^{-t}; s\Big\} = \frac{1}{s+1}$$

So, $s\,F(s) = \dfrac{s}{s+1}$

$\therefore \lim_{s\to 0} s\,F(s) = 0$ and $\lim_{t\to\infty} f(t) = 0$.

So, $\lim_{s\to 0} s\,F(s) = \lim_{t\to\infty} f(t)$. Hence the result.

1.6 The Heaviside Step Function

The Heaviside step function is denoted and defined by

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a,\ a \geq 0 \end{cases}$$

where a is a real number, and is depicted in Figure 1.2.

The Laplace transform of $H(t-a)$ is given by

$$\begin{aligned} L\Big\{H(t-a)\Big\} &= \int_0^\infty e^{-st}H(t-a)dt = \int_0^a e^{-st}H(t-a)dt + \int_a^\infty e^{-st}H(t-a)dt \\ &= \int_0^a e^{-st}\cdot 0\,dt + \lim_{X\to\infty}\int_a^X e^{-st}\cdot 1\,dt = 0 + \lim_{X\to\infty}\left[\frac{e^{-st}}{-s}\right]_a^X = \frac{e^{-as}}{s} \end{aligned}$$

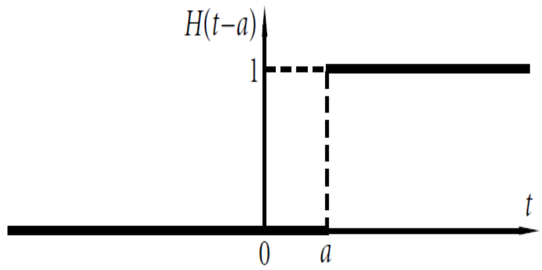

Figure 1.2: Heaviside step function

Property 1.4 (Second Shifting property:) If $L\Big\{f(t)\Big\} = F(s)$, Prove that

$$L\Big\{f(t-a)H(t-a)\Big\} = e^{-as}F(s), s > a > 0.$$

Proof.

$$\begin{aligned} L\Big\{f(t-a)\cdot H(t-a)\Big\} &= \int_0^\infty e^{-st} f(t-a)\cdot H(t-a)\, dt \\ &= \int_{-a}^\infty e^{-s(x+a)} f(x)H(x)\, dx, \text{taking } t-a=x \\ &= e^{-as}\int_0^\infty e^{-st} f(t)\, dt = e^{-as}F(s) \end{aligned}$$

Example 1.12 The Heaviside step function is very useful in dealing with functions with discontinuities or piecewise smooth functions. The following are some examples:(i - iv)

(i) $f(t) = \begin{cases} f_1(t), & t < t_0 \\ 0, & t \geq t_0 \end{cases}$

$= f_1(t)\Big[1 - H(t-t_0)\Big]$

(ii) $f(t) = \begin{cases} 0, & t < t_0 \\ f_2(t), & t \geq t_0 \end{cases}$

$= f_2(t)H(t-t_0)$

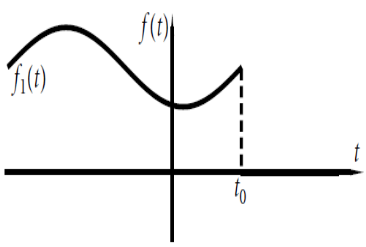

Figure 1.3: Graph of function i

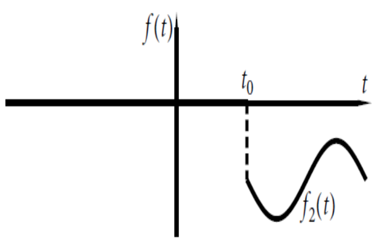

Figure 1.4: Graph of function ii

$$\textbf{(iii)}\ f(t) = \begin{cases} f_1(t), & t < t_0 \\ f_2(t), & t \geq t_0 \end{cases} \qquad \textbf{(iv)}\ g(t) = \begin{cases} 0, & t < a \\ g(t), & a \leq t < b \\ 0, & t \geq b \end{cases}$$

$$= \begin{cases} f_1(t) + \Big[f_2(t) - f_1(t)\Big] \cdot 0, & t < t_0 \\ f_1(t) + \Big[f_2(t) - f_1(t)\Big] \cdot 1, & t \geq t_0 \end{cases} \qquad = G(t)\Big[H(t-a) - H(t-b)\Big]$$

$$= f_1(t) + \Big[f_2(t) - f_1(t)\Big]H(t - t_0)$$

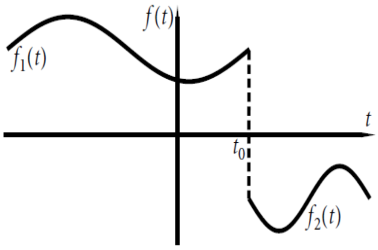

Figure 1.5: Graph of function iii

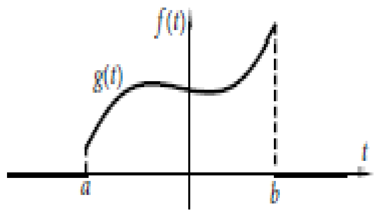

Figure 1.6: Graph of function iv

If function f has nonzero values only in the range of $a < t < b$ as shown in the following Figure-1.6, than it can be expressed as since

$$H(t-a) - H(t-b) = \begin{cases} 0, & t < a \\ 1, & a \leq t < b \\ 0, & t \geq b \end{cases}$$

As a generalization, function f of the following form can be easily written in terms of the Heaviside step function

$$\begin{aligned} f(t) &= \begin{cases} 0, & t < t_0 \\ f_1, & t_0 \leq t < t_1 \\ f_2, & t_1 \leq t < t_2 \\ \cdots & \\ f_n, & t_{n-1} \leq t < t_n \\ 0, & t \geq t_n \end{cases} \\ &= f_1(t)\Big[H(t-t_0) - H(t-t_1)\Big] + f_2(t)\Big[H(t-t_1) - H(t-t_2)\Big] + \cdots + \\ &\quad + f_n(t)\Big[H(t-t_{n-1}) - H(t-t_n)\Big] \end{aligned}$$

1.7 Some Important Results

$f(t)$	$L\left\{f(t)\right\} = F(s)$	$f(t)$	$L\left\{f(t)\right\} = F(s)$
1	$\frac{1}{s}, s > 0$	t	$\frac{1}{s^2}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}, s > 0, n > -1$
$\sin at$	$\frac{a}{s^2+a^2}, \lvert s\rvert > a$	$\cos at$	$\frac{s}{s^2+a^2}, \lvert s\rvert > a$
$\sinh at$	$\frac{a}{s^2-a^2}, \lvert s\rvert > a$	$\cosh at$	$\frac{s}{s^2-a^2}, \lvert s\rvert > a$

1.8 Worked Out Examples

Example 1.13 Find $L[f(t)]$,

$$\text{where } f(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t \le \frac{\pi}{3} \end{cases}$$

Solution: Using the second shifting property of LT,

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t \le a \end{cases}$$

$$\begin{aligned} \text{Then, } L\left\{g(t)\right\} &= e^{-as}F(s), \text{ where } L\left\{f(t)\right\} = F(s), \text{ here } a = \frac{\pi}{3} \\ &= e^{-\frac{\pi}{3}s}L\left\{\sin t\right\} = e^{-\frac{\pi}{3}s}\frac{1}{s^2+1} \end{aligned}$$

Example 1.14 Find $L[f(t)]$, where $f(t) = \begin{cases} e^{-t}, & 0 \le t < 2 \\ 0, & \text{otherwise} \end{cases}$

Solution: By the definition of LT

$$\begin{aligned} L\left\{f(t)\right\} &= \int_0^\infty e^{-st}f(t)dt \\ &= \int_0^2 e^{-st}e^{-t}dt = \int_0^2 e^{-(1+s)t}dt \\ &= \left[-\frac{e^{-(1+s)t}}{1+s}\right]_0^2 = \frac{1-e^{-2(1+s)}}{1+s}. \end{aligned}$$

Example 1.15 Find $L[f(t)]$,

$$\textit{where } f(t) = \begin{cases} 0, & t < 1 \\ t, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$$

Solution: By the definition of LT

$$\begin{aligned} L\Big\{f(t)\Big\} &= \int_0^\infty e^{-st} f(t)dt \\ &= \int_1^2 e^{-st} t dt = \left[t\frac{e^{st}}{s} - \frac{e^{-st}}{s^2}\right]_1^2 \\ &= \frac{(s+1)e^{-s} - (2s+1)e^{-2s}}{s^2} \end{aligned}$$

Example 1.16 Using the linearity of the Laplace transform, calculate the Laplace transform of

$$f(t) = \sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

$$\begin{aligned} \textbf{Solution: } L(\sinh(at)) &= L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at}) \\ &= \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} \\ &= \frac{1}{2}\frac{s+a-(s-a)}{s^2-a^2} = \frac{a}{s^2-a^2} \end{aligned} \tag{1.9}$$

Example 1.17 Using the shift theorem find the Laplace transform of

$$f(t) = e^{2t}t^2$$

Solution: By the first shift theorem,

$$L\left(e^{-at}f(t)\right) = F(s-a) \tag{1.10}$$

where $L(f) = F(s)$. Now, we know that

$$L\left(t^2\right) = \frac{2!}{s^3} = \frac{2}{s^3} \tag{1.11}$$

so, by the shift theorem

$$L\left(e^{2t}t^2\right) = \frac{2}{(s-2)^3} \tag{1.12}$$

Example 1.18 Find the Laplace transform

$$y'' + 4y' + 8y = \cos 2t \tag{1.13}$$

Given that $y = 2$ and $y' = 1$ when $t = 0$.

Solution: Applying Laplace transform in both sides with respect to t in the equation (1.13), we obtain $\{s^2Y(s) - sy(0) - y'(0)\} + 4\{sY(s) - y(0)\} + 8Y(s) = \frac{s}{s^2+4}$. Using the initial conditions, we get, $s^2Y(s) - 2s - 1 + 4sY(s) - 8 + 8Y(s) = \frac{s}{s^2+4}$ or $(s^2 + 4s + 8)Y(s) = \frac{s}{s^2+4} + 2s + 9$.

$$\begin{aligned} \text{Therefore } Y(s) &= \frac{1}{20} \times \frac{s}{s^2+4} + \frac{1}{5} \times \frac{1}{s^2+4} - \frac{1}{20} \times \frac{(s+2)-2}{(s+2)^2+2^2} \\ &- \frac{2}{5} \times \frac{1}{(s+2)^2+2^2} + \frac{2(s+2)-4}{(s+2)^2+2^2} + \frac{9}{(s+2)^2+2^2} \end{aligned}$$

Example 1.19 Find the Laplace transform of both side of the identity

$$\frac{d}{dt}\cosh 3t = 3\sinh 3t$$

and verify that you get the same answer on each side. The idea is that you do the right hand side using the table entry for $\sinh(3t)$ and the left hand side using the formula for f' with $f = \cosh(3t)$. $\cosh(0) = 1$ by the way.

Solution: We know that

$$L(\sinh at) = \frac{a}{s^2 - a^2}, \qquad L(\cosh at) = \frac{s}{s^2 - a^2} \tag{1.14}$$

and $\cosh 0 = 1$ so

$$\begin{aligned}
&L\left(\frac{d}{dt}\cosh 3t\right) = L\left(3\sinh 3t\right)\\
&sL\left(\cosh 3t\right) - 1 = 3\frac{3}{s^2-9}\\
&s\frac{s}{s^2-9} - 1 = 3\frac{3}{s^2-9}\\
&s\frac{s}{s^2-9} - \frac{s^2-9}{s^2-9} = \frac{9}{s^2-9}\\
&\frac{9}{s^2-9} = \frac{9}{s^2-9}
\end{aligned} \tag{1.15}$$

Example 1.20 Find the Laplace transform of both sides of the differential equation

$$2\frac{df}{dt} = 1$$

with initial conditions $f(0) = 4$. By solving the resulting equations find $F(s)$. Based on the Laplace trasforms you know, decide what $f(t)$ is?

Solution: Using linearity of L, plus the property of Laplace transforms of derivatives, we get

$$\begin{aligned}
&L\left(2\frac{df}{dt}\right) = L(1) \Rightarrow 2L\left(\frac{df}{dt}\right) = \frac{1}{s}\\
&2sF(s) - 8 = \frac{1}{s} \qquad (1.16)\\
\text{This means that}\quad &F(s) = \frac{4}{s} + \frac{1}{2s^2}\\
\text{and, since,}\quad &L(t^n) = n!/s^{n+1}\\
&\therefore\ f(t) = 4 + \frac{1}{2}t \qquad (1.17)
\end{aligned}$$

Example 1.21 Find the convolution $(f * g)(t)$ when $f(t) = t$, $g(t) = e^{2t}$ $(t \geq 0)$.

Solution: From the definition of convolutions

$$\begin{aligned}(f*g)(t) &= \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t \tau e^{2(t-\tau)}\,d\tau \\ &= \int_0^t \tau e^{2t}e^{-2\tau}\,d\tau = e^{2t}\int_0^t \tau e^{-2\tau}\,d\tau\end{aligned}$$

Let $u = \tau, \quad dv = e^{-2\tau}\,d\tau$ then $du = d\tau, \quad v = -\frac{1}{2}e^{-2\tau}$. Use integration by parts, we get,

$$\begin{aligned}& e^{2t}\int_0^t u\,dv = e^{2t}\left([uv]_0^t - \int_0^t v\,du\right) \\ =\ & e^{2t}\left(\left[-\frac{\tau}{2}e^{-2\tau}\right]_0^t - \int_0^t -\frac{1}{2}e^{-2\tau}\,d\tau\right) \\ =\ & e^{2t}\left(-\frac{t}{2}e^{-2t} + 0 + \frac{1}{2}\int_0^t e^{-2\tau}\,d\tau\right) \\ =\ & -\frac{t}{2} + \frac{e^{2t}}{2}\left[-\frac{1}{2}e^{-2\tau}\right]_0^t \\ =\ & -\frac{t}{2} + \frac{e^{2t}}{2}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right) \\ =\ & -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t}\end{aligned}$$

Example 1.22 Use the convolution theorem to find the function f with

$$L(f) = \frac{1}{s^2(s-4)}. \tag{1.18}$$

Solution: We know $L(t)) = \frac{1}{s^2}$ and $L(e^{4t}) = \frac{1}{s-4}$. From the convolution theorem, we see

$$L(f) = \frac{1}{s^2(s-4)} = L(t)L(e^{4t}) = L(t * e^{4t})$$

so that $f(t)$ is the convolution $t * e^{4t}$.

$$\begin{aligned}f(t) &= \int_0^t \tau e^{4(t-\tau)}\,d\tau \\ &= \int_0^t \tau e^{4t}e^{-4\tau}\,d\tau = e^{4t}\int_0^t \tau e^{-4\tau}\,d\tau\end{aligned}$$

Let $U = \tau, \quad dV = e^{-4\tau}\, d\tau,$ then $dU = d\tau, \quad V = -\frac{1}{4}e^{-4\tau}$. Use integration by parts, we get

$$e^{4t}\int_0^t U\,dV = e^{4t}\left([UV]_0^t - \int_0^t V\,dU\right)$$

$$\begin{aligned}
&= e^{4t}\left(\left[-\frac{\tau}{4}e^{-4\tau}\right]_0^t - \int_0^t -\frac{1}{4}e^{-4\tau}\,d\tau\right)\\
&= e^{4t}\left(-\frac{t}{4}e^{-4t} + 0 + \frac{1}{4}\int_0^t e^{-4\tau}\,d\tau\right)\\
&= -\frac{t}{4} + \frac{e^{4t}}{4}\left[-\frac{1}{4}e^{-4\tau}\right]_0^t\\
&= -\frac{t}{4} + \frac{e^{4t}}{4}\left(-\frac{1}{4}e^{-4t} + \frac{1}{4}\right)\\
&= -\frac{t}{4} - \frac{1}{16} + \frac{1}{16}e^{4t}
\end{aligned}$$

Example 1.23 Find the Laplace transform to one dimensional Heat equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ with

BCS: $u(0,t) = 0, \qquad u(5,t) = 0,\ t > 0$

IC: $u(x,0) = 10\sin 4\pi x,\ 0 < x < 5$

Solution: Applying Laplace transform in both sides in the equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, we get

$$\begin{aligned}
&L\left\{\frac{\partial u}{\partial t}\right\} = 2L\left\{\frac{\partial^2 u}{\partial x^2}\right\}\\
&\Rightarrow 2\frac{d^2U(x,s)}{dx^2} = sU(x,s) - u(x,0)\\
&\Rightarrow 2\frac{d^2U(x,s)}{dx^2} = sU(x,s) - 10\sin 4\pi x\\
&\Rightarrow \frac{d^2U(x,s)}{dx^2} - \frac{s}{2}U(x,s) = -5\sin 4\pi x
\end{aligned} \tag{1.19}$$

$\therefore$ the Laplace transform of the given heat equation is $\frac{d^2U(x,s)}{dx^2} - \frac{s}{2}U(x,s) = -5\sin 4\pi x$.

Example 1.24 Find the Laplace transform

$$\frac{dx}{dt} - y = e^t,\ \ \frac{dy}{dt} + x = \sin t.\ \text{Given that } x(0) = 1 \text{ and } y(0) = 0. \tag{1.20}$$

Solution: Applying Laplace transform in both equations and using the notations $X = L[x;s]$, $Y = L[y;s]$, we get, $sX - x(0) - Y = \frac{1}{s-1}$, $sY - y(0) + X = \frac{1}{s^2+1}$. Using the given initial conditions, the Laplace transform of the given equations is

$$sX - Y = \frac{s}{s-1} \tag{1.21}$$

$$X + sY = \frac{1}{s^2+1} \tag{1.22}$$

1.9 Multiple Choice Questions(MCQ)

1. Given that $F(S)$ is the one side Laplace transformation of $f(t)$, then Laplace transformation of $L\left\{\int_0^t f(\tau)d\tau\right\}$ is equals to **GATE(CE)-2009**
(A) $sF(s) - f(0)$ (B) $\frac{1}{s}F(s)$ (C) $\int_0^s f(\tau)d\tau$ (D) $\frac{1}{s}F(s) - f(0)$
Ans. (B)
2. If Laplace transformation of $f(t) = \frac{1-e^t}{t}$ is equals to
(A) $\log\left(\frac{s-1}{s}\right)$ (B) $\log\left(\frac{s+1}{s}\right)$ (C) $\log\left(\frac{s}{s-1}\right)$ (D) $\log\left(\frac{s}{s+1}\right)$
Ans. (A)

$$\textbf{Hint. } L\left\{1-e^t\right\} = L\left\{1\right\} - L\left\{e^t\right\} = \frac{1}{s} - \frac{1}{s-1} = F(s) \text{ (say)}$$
$$L\left\{\frac{1-e^t}{t}\right\} = \int_0^s F(s)ds$$
$$= \int_0^s \left(\frac{1}{s} - \frac{1}{s-1}\right)ds = \log\left(\frac{s-1}{s}\right)$$

3. If Laplace transformation of $L\left\{\int_0^t \frac{\sin t}{t}\right\}$ is equals to
(A $\frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}s\right)$ (B)$\frac{1}{s}\left(\frac{\pi}{2} + \tan^{-1}s\right)$ (C) $\left(\frac{\pi}{2} - \tan^{-1}s\right)$ (D) $\frac{1}{s}\left(\frac{\pi}{2} - \cot^{-1}s\right)$
Ans. (A)
Hint. Since

$$L\left\{\sin t\right\} = \frac{1}{s^2+1}$$
$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}ds = \frac{\pi}{2} + \tan^{-1}s = F(s) \text{ (say)}$$
$$L\left\{\int_0^t \frac{\sin t}{t}\right\} = \frac{F(s)}{s} = \frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}s\right)$$

4. If $L\left\{J_0(t)\right\} = \frac{1}{\sqrt{s^2+1}}$, then $L\left\{J_0(5t)\right\}$ is equals to
(A $\frac{1}{\sqrt{s^2+25}}$ (B) $\frac{5}{\sqrt{s^2+25}}$ (C) $\frac{1}{\sqrt{s^2-25}}$ (D)$\frac{s}{\sqrt{s^2+25}}$
Ans. (A)
Hint. If $L\left\{f(t)\right\} = F(s)$, then

$$L\left\{f(at)\right\} = \frac{1}{s}F\left\{\frac{s}{a}\right\}$$
$$L\left\{J_0(5t)\right\} = \frac{1}{5}F\left\{\frac{s}{5}\right\} = \frac{1}{\sqrt{s^2+25}}$$

5. Laplace transformation of f(t) , where

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

is equals to

(A) $e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2}$ (B) $e^{-s}\frac{1}{s^2} + 2e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2}$
(C) $e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} - e^{-3s}\frac{1}{s^2}$ (D) $e^{-s}\frac{1}{s^2} - 2e^{2s}\frac{1}{s^2} + e^{3s}\frac{1}{s^2}$
Ans. (A) Since

$$\begin{aligned} L\Big\{f(t)\Big\} &= (t-1)\Big\{H(t-1) - H(t-2)\Big\} + (3-t)\Big\{H(t-2) + H(t-3)\Big\} \\ &= (t-1)\Big\{H(t-1)\Big\} - 2(t-2)\Big\{H(t-2)\Big\} + (t-3)\Big\{H(t-3)\Big\} \\ &= e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2} \end{aligned}$$

6. The Laplace transformation of $(t^2 - 2t)H(t-1)$ is **GATE(EE)-98**
(A) $\frac{2e^{-s}}{s^3} - \frac{2e^{-s}}{s^2}$ (B) $\frac{2e^{-2s}}{s^3} - \frac{2e^{-s}}{s^2}$
(C) $\frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2}$ (D) $\frac{2e^{-2s}}{s^3} - \frac{e^{-s}}{s}$
Ans. (D)
Hint. Since $L\Big\{f(t-a)H(t-a)\Big\} = e^{-as}F(s)$. Therefore

$$\begin{aligned} L\Big\{(t^2-2t)H(t-1)\Big\} &= L\Big\{\Big((t-1)^2 - 1\Big)H(t-1)\Big\} \\ &= L\Big\{\Big((t-1)^2\Big)H(t-1)\Big\} - L\Big\{H(t-1)\Big\} = e^{-s}\frac{2}{s^3} - \frac{e^{-s}}{s} \end{aligned}$$

7. Consider the function $\frac{5}{s(s^2+3s+2)}$, where $F(s)$ is the Laplace transformation of the function f the initial value of $f(t)$ is equals to **GATE(EE)-04**
(A) 5 (B) $\frac{5}{2}$ (C) $\frac{5}{3}$ (D) 0
Ans. (D)
Hint. By applying initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{5}{s(s^2+3s+2)}\right) = 0$$

8. Consider the function $\frac{s}{s^2+6s+2}$, where $F(s)$ is the Laplace transformation of the function f the initial value of $f(t)$ is equals to
(A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 0
Ans. (C)
Hint. By applying initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s^2}{s^2+6s+2} = 1$$

9. If Laplace transformation of $f(t)$ is $\frac{5}{s} + \frac{2s}{s^2+9}$, Then $f(0)$ is equals to
(A) 5 (B) 7 (C) 0 (D) ∞
Ans. (B)
Hint. By applying initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{5}{s} + \frac{2s}{s^2+9}\right) = 5 + 2 = 7$$

10. If Laplace transformation of $f(t)$ is $F(s) = \frac{2}{s(s+1)}$. Then $f(\infty)$ is equals to

A) 0 (B) 2 (C) 1 (D) $\propto$ **GATE(ECE)-03**

Ans. (B)

Hint. By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{2}{s(s+1)}\right) = 2$$

11. If $F(s) = \frac{2(s+1)}{s^2+4s+7}$. Then the initial and final values of $f(t)$ are respectively **GATE(ECE)-11**

(A) $0, 2$ (B) $2, 0$ (C) $0, 2/7$ (D) $2/7, 0$

Ans. (B)

Hint. By applying initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{2(s+1)}{s^2+4s+7}\right) = 2$$

By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{2(s+1)}{s^2+4s+7}\right) = 0$$

12. If $L[f(t)] = \frac{k}{s(s^2+4)}$. If $\lim_{t\to\infty} f(t) = 1$, then k is given by

(A) 4 (B) zero (C) $0 < k < 12$ (D) $5 < k < 12$

Ans. (A)

Hint. By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \Rightarrow \lim_{s\to 0} s\left(\frac{k}{s(s^2+4)}\right) = 1 \Rightarrow k = 4.$$

13. The Laplace transformation of $f(t)$ is given by $F(s) = \frac{2}{s(s+1)}$. As $t \longrightarrow \infty$, the value of $f(t)$ tends to **ECE-2003**

(A) 0 (B) 1 (C) 2 (D) ∞

Ans. (C)

Hint. By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{2}{s(s+1)}\right) = 2$$

14. Use Laplace transformation the value of $\int_0^\infty te^{-2t}\sin t\,dt$ is

A) $\frac{1}{25}$ B) $\frac{2}{25}$ C) $\frac{3}{25}$ D) $\frac{4}{25}$

Ans. (D)

Hint. Since $L\{\sin t\} = \frac{1}{s^2+1}$ and $L\{t\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \overline{f}(s)$. Now from the definition of Laplace transformation

$$\int_0^\infty e^{-st} f(t)dt = \overline{f}(s) \Rightarrow \int_0^\infty t\sin te^{-2t}dt = \overline{f}(2) = \frac{2\times 2}{(2^2+1)^2} = \frac{4}{25}$$

15. Let y be the solution of the initial value problem

$$\frac{d^2y}{dx^2} + y = 6\cos 2x, \quad y(0) = 3, \quad y^{'} = 1$$

Let the Laplace transformation of y be $F(s)$. Then the value of $F(1)$ is **GATE(MA)-11**
A) $\frac{17}{5}$ B) $\frac{13}{5}$ C) $\frac{11}{5}$ D) $\frac{9}{5}$
Ans. B)
Hint. Applying Laplace transform in both sides with respect to t in the equation (**??**), we obtain $\{s^2F(s) - sy(0) - y'(0)\} + F(s) = \frac{6s}{s^2+4}$. Using the initial conditions, we get, $s^2F(s) - 3s - 1 + F(s) = \frac{6s}{s^2+4}$, $(s^2+1)F(s) = 3s + 1 + \frac{6s}{s^2+4}$. Therefore, $F(1) = \frac{13}{5}$.

16. If $Y(s)$ is the Laplace transform of $y(t)$ which is the solution of the initial value problem

$$\frac{d^2y}{dx^2} + y(t) = \begin{cases} 0, & 0 < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$$

, with $y(0) = 1$ and $y^{'}(0) = 0$, then $Y(s)$ equals **GATE(MA)-04**
A) $\frac{s}{1+s^2} + \frac{e^{-2\pi s}}{(1+s^2)^{\frac{3}{2}}}$ B) $\frac{s+1}{1+s^2}$ C) $\frac{s}{1+s^2} + \frac{e^{-2\pi s}}{(1+s^2)}$ D) $\frac{s(1+s^2)+1}{(1+s^2)^2}$
Ans. A)
Hint.

$$\frac{d^2y}{dx^2} + y(t) = \begin{cases} 0, & 0 < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$$

Taking Laplace in both sides

$$p^2y(s) - sy(0) - y^{'}(0) + y(s) = \int_{2\pi}^{\infty} e^{-pt}\sin t dt$$
$$\Rightarrow (s^2+1)y(s) - s = 0 - \frac{e^{-2\pi s}}{\sqrt{1+s^2}}(0-1) \Rightarrow y(s) = \frac{s}{1+s^2} + \frac{e^{-2\pi s}}{(1+s^2)^{\frac{3}{2}}}$$

17. Given that the Laplace transform, $L\Big\{e^{at}f(t)\Big\} = F(s-a)$, $s > a$, then $L\Big\{3e^{5t}\sin 5t\Big\} =$
A) $\frac{3s}{s^2-10}$ A) $\frac{15}{s^2-10s}$ C) $\frac{3s}{s^2+10s}$ D) $\frac{15}{(s-5)^2+25}$ **GATE(AE)-2013**
Ans. (D)

Hint. $L\Big\{\sin 5t\Big\} = \frac{5}{s^2+25}$ so $L\Big\{3e^{5t}\sin 5t\Big\} = 3 \times \frac{5}{(s-5)^2+25} = \frac{15}{(s-5)^2+25}$

18. Given that the Laplace transform, $L\Big\{e^{at}f(t)\Big\} = F(s-a)$, $s > a$, then $L\Big\{e^{3t}\cos 3t\Big\} =$
A) $\frac{s-3}{(s-3)^2+9}$ A) $\frac{s}{(s-3)^2+9}$ C) $\frac{3}{(s-3)^2+9}$ D) $\frac{s-9}{(s-3)^2+16}$
Ans. (A)

Hint. $L\Big\{\cos 3t\Big\} = \frac{s}{s^2+9}$ so $L\Big\{e^{3t}\cos 3t\Big\} = \frac{s-3}{(s-3)^2+9}$

1.10 Review Exercise

1. Prove that $L\left\{e^t \cos t \sin t\right\} = \frac{1}{s^2-2s+5}$

2. If $L\left\{f(t)\right\} = \frac{s^2-s+1}{(s-1)(2s+1)^2}$, then prove that by change of scale property
$$L\left\{f(2t)\right\} = \frac{s^2-2s+4}{4(s+1)^2(s-1)}$$

3. Use the formula for the Laplace transform of a periodic function with period c: $L(f) = \frac{1}{1-e^{-cs}}\int_0^c f(t)e^{-st}dt$. Prove that $-\frac{1}{s}\left[-\frac{1}{s}(e^{-\pi s}+1)+\frac{1}{s}I\right]$ is the Laplace transform of a half-rectified wave $f(t) = \begin{cases} \sin t & \sin t > 0 \\ 0 & \sin t \leq 0 \end{cases}$

4. Prove that
$$L(\cosh at \cosh bt) = \frac{s[(s^2-a^2-b^2]}{(s^2-(a-b)^2)(s^2-(a+b)^2)}$$

5. Find the Laplace transform of both sides of the differential equation $2\frac{df}{dt} = 1$ with initial conditions $f(0) = 4$.
Ans. $F(s) = \frac{4}{s} + \frac{1}{2s^2}]$

6. Find the Laplace transform of both sides of the differential equation $y'' - 4y' + 3y = 6t - 8$ with initial conditions $y(0) = y'(0) = 0$.
Ans. $Y = \frac{6}{s^2(s^2-4s+3)} - \frac{8}{s(s^2-4s+3)}$

7. Find the Laplace transform $t\,y'' + y' + ty = 0$ with $y = 1$ and $y' = 0$ when $t = 0$.
Ans. $(s^2+1)\frac{dY(s)}{ds} + sY(s) = 0]$.

8. Find the Laplace transform of $J_0(t)$ by using IVP.
Ans. $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$.
Hint. $J_n(t)$ is the solution of Bessel function of order n i.e. $t^2\frac{d^2J_n(t)}{dt^2} + t\frac{dJ_n(t)}{dt} + (t^2-n^2)J_n(t) = 0$.

9. Find the Laplace transform of $tJ_1(t)$ by using IVP. **Ans.** $L\{tJ_1(t)\} = \frac{1}{(s^2+1)^{\frac{3}{2}}}$.

10. Using Laplace transform, show that $\int_0^\infty te^{-3t}\sin t dt = \frac{3}{50}$.

11. Show that $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = \frac{3s-24}{s^2+4s+40}$.

12. Given that $L\{\frac{\sin t}{t}\} = \tan^{-1}(\frac{1}{s})$, find $L\{\frac{\sin at}{t}\}$.
Ans. $L\{\frac{\sin at}{t}\} = \tan^{-1}(\frac{a}{s})$.

13. If $f(t) = \int_0^t \frac{g(u)}{u}du$, show that $L\{f(t)\} = \frac{1}{s}\int_s^\infty G(u)du$ where $G(u) = \int_0^\infty g(u)e^{-ut}dt$.

14. If $f(t) = \int_t^\infty \frac{g(u)}{u}du$, show that $L\{f(t)\} = \frac{1}{s}\int_0^s G(u)du$ where $G(u) = \int_0^\infty g(u)e^{-ut}dt$.

15. Evaluate $L\{\int_0^t \frac{\sin u}{u}du\}$ by the help of Initial Value Theorem.

Ans. $L\{\int_0^t \frac{\sin u}{u}du\} = \frac{1}{s}\tan^{-1}(\frac{1}{s})$.

Chapter 2

Inverse Laplace Transformation

2.1 Introduction

In this chapter, we have presented the formal definition of inverse Laplace transform. The basic operational properties of the inverse Laplace transforms including convolution theorem are discussed in detail. Also, the solution of the differential equation can be obtained by determining the inverse Laplace transform. The chapter is systematically developed with proper examples.

Definition 2.1 Inverse Laplace Transformation: If Laplace transform $F(s)$ of the function f in t, i.e $F(s) = L\left\{f(t)\right\}$, the inverse Laplace transform is $f(t) = L^{-1}\left\{F(s)\right\}$ where $L^{-1}\left\{F(s)\right\}$ is the inverse operator of the Laplace transform i.e. it restores the Laplace transform to the original function. Examples are,

$$\begin{aligned}
L^{-1}\left\{\frac{1}{s-a}\right\} &= e^{at}, \qquad L^{-1}\left\{\frac{1}{s}\right\} = 1 \\
L^{-1}\left\{\frac{a}{s^2+a^2}\right\} &= \sin at, \qquad L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at \\
L^{-1}\left\{\frac{a}{s^2-a^2}\right\} &= \sinh at, \qquad L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at \\
L^{-1}\left\{\frac{1}{s^{n+1}}\right\} &= \frac{t^n}{\Gamma(n+1)}, \qquad L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at
\end{aligned}$$

Definition 2.2 Null Functions and Uniqueness We define a null function N in t as one for which $\int_0^T N(t)dt = 0$ for all positive T. A null function cannot be a continuous function unless it vanishes for all $t \geq 0$. A theorem due to **Lerch** states that if $L\{f(t)\} = F(s) = L\{g(t)\}$ then $f(t) - g(t) = N(t)$.
Consequently, given $F(s)$ for which we find a *continuous* inverse $f(t)$ over a given closed interval, this function is the *unique continuous solution* for the inverse over that interval.

2.2 Properties of Inverse Laplace transformation

Property 2.1 Linear property of Inverse Laplace Transformation:
If $L^{-1}\left\{F_1(s)\right\} = f_1(t)$, and $L^{-1}\left\{F_2(s)\right\} = f_2(t)$, then $L^{-1}\left\{c_1F_1(s) + c_2F_2(s)\right\} = c_1f_1(t) + c_2f_2(t)$.

Proof: Since $L^{-1}\Big\{F_1(s)\Big\} = f_1(t)$, and $L^{-1}\Big\{F_2(s)\Big\} = f_2(t)$, then $L\Big\{f_1(t)\Big\} = F_1(s)$ and $L\Big\{f_2(t)\Big\} = F_2(s)$.

$$\text{So}, L\Big\{c_1 f_1(t) + c_2 f_2(t)\Big)\Big\} = c_1 L\Big\{f_1(t)\Big)\Big\} + c_2 L\Big\{f_2(t)\Big)\Big\} = c_1 F_1(s) + c_2 F_2(s)$$

$$\Rightarrow L^{-1}\Big\{c_1 F_1(s) + c_2 F_2(s)\Big\} = c_1 f_1(t) + c_2 f_2(t) \qquad \textbf{(Hence proved)}$$

Property 2.2 Shifting property of Inverse Laplace Transformation:
If $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L^{-1}\Big\{F(s-a)\Big\} = e^{at} f\Big(t\Big)$

Proof: Since $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L\Big\{f(t)\Big\} = F(s)$. By first shifting property, we know that

$$L\Big\{e^{at} f\Big(t\Big)\Big\} = F(s-a) \Rightarrow L^{-1}\Big\{F(s-a)\Big\} = e^{at} f\Big(t\Big) \qquad \textbf{(Hence proved)}$$

Property 2.3 Change of scale of Inverse Laplace Transformation:
If $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L^{-1}\Big\{F(as)\Big\} = \frac{1}{a} f\Big(\frac{t}{a}\Big)$

Proof: Since $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L\Big\{f(t)\Big\} = F(s)$

$$L\Big\{\frac{1}{a} f\Big(\frac{t}{a}\Big)\Big\} = \frac{1}{a} L\Big\{f\Big(\frac{t}{a}\Big)\Big\} = \frac{1}{a} \frac{1}{\frac{1}{a}} F\Big(\frac{s}{\frac{1}{a}}\Big) = F(as)$$

$$\text{Hence}, L^{-1}\Big\{F(as)\Big\} = \frac{1}{a} f\Big(\frac{t}{a}\Big) \qquad \textbf{(Hence proved)}$$

2.3 Operational Rules of Inverse Laplace Transforms

Theorem 2.1 Multiplication by s^n:
If $L^{-1}\Big\{F(s)\Big\} = f(t)$, and $f(0) = 0$, then $L^{-1}\Big\{sF(s)\Big\} = f'(t)$

Proof: Since $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L\Big\{f(t)\Big\} = F(s)$. Therefore

$$L\Big\{f'(t)\Big\} = sF(s) - f(0) = sF(s), \qquad \text{Since } f(0) = 0$$

$$\Rightarrow L^{-1}\Big\{sF(s)\Big\} = f'(t) \qquad \textbf{(Hence proved)}$$

Example 2.1 Find

$$L^{-1}\Big\{\frac{s}{(s+1)^5}\Big\}.$$

Solution:

$$\text{Since } L^{-1}\Big\{F(s)\Big\} = f(t)$$

$$\text{So, } L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s+1)^5}\right\} = e^{-t}\frac{t^4}{4!}$$

$$\therefore f(t) = e^{-t}\frac{t^4}{4!}$$

Also we have $L\Big\{f'(t)\Big\} = sF(s) - f(0)$

$$\text{So, } L^{-1}\left\{\frac{s}{(s+1)^5}\right\} = \frac{d}{dt}\left\{e^{-t}\frac{t^4}{4!}\right\}, \because f(0) = 0$$

$$= \frac{1}{4!}(e^{-t}4t^3 - e^{-t}t^4) = \frac{e^{-t}t^3}{24}(4-t)$$

Theorem 2.2 Division by s:
If $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)du$

Proof: Let $\phi(t) = \int_0^t f(u)du$. Therefore $\phi'(t) = f(t)$ and $\phi(0) = 0$. If $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L\Big\{f(t)\Big\} = F(s)$. Therefore

$$L\Big\{\phi'(t)\Big\} = s\Phi(s) - \phi(0) = s\Phi(s)$$

$$\Rightarrow L\Big\{f(t)\Big\} = s\Phi(s) \Rightarrow \frac{F(s)}{s} = L\Big\{\phi(t)\Big\}$$

$$\Rightarrow L^{-1}\left\{\frac{F(s)}{s}\right\} = \phi(t) = \int_0^t f(u)du \qquad \textbf{(Hence proved)}$$

Theorem 2.3 Inverse Laplace Transformation of Integral:
If $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L^{-1}\left\{\int_s^\infty F(u)du\right\} = \frac{f(t)}{t}$

Proof: Since $L^{-1}\Big\{F(s)\Big\} = f(t)$, then $L\Big\{f(t)\Big\} = F(s)$. Therefore

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$$

$$\Rightarrow L^{-1}\left\{\int_s^\infty F(u)du\right\} = \frac{f(t)}{t} \qquad \textbf{(Hence proved)}$$

Theorem 2.4 Convolution Theorem:
If $L^{-1}\Big\{F(s)\Big\} = f(t)$, and $L^{-1}\Big\{G(s)\Big\} = g(t)$ then $L^{-1}\Big\{F(s)\cdot G(s)\Big\} = f(t) \star g(t) = \int_0^u f(u-t)g(t)dt$

Proof. From the definition of the Laplace transform, we know that

$$\begin{aligned} F(s)\,G(s) &= [\int_0^\infty f(t)e^{-st}dt]\,[\int_0^\infty g(v)e^{-sv}dv] \\ &= \int_0^\infty \int_0^\infty e^{-s(t+v)} f(t)g(v)dt\,dv \\ &= \int_0^\infty g(t)\{\int_0^\infty e^{-(t+v)} f(v)\,dv\}\,dt \end{aligned}$$

Let $t + v = u$ in the inner integral. Then

$$F(s)\,G(s) = \int_0^\infty g(t)\{\int_t^\infty e^{-su} f(u-t)\,du\}\,dt$$

Then we have $F(s)\,G(s) = \int_0^\infty \{\int_0^u e^{-su} f(u-t)g(t)\,dt\}\,du$

$$= \int_0^\infty e^{-su}\{\int_0^u f(u-t)g(t)\,dt\}\,du \;= L[\int_0^u f(u-t)g(t)\,dt; u]$$

Using inverse Laplace transform, we get $L^{-1}\Big\{F(s)\cdot G(s)\Big\} = f(t) \star g(t) = \int_0^u f(u-t)g(t)dt$

2.4 Worked Out Examples

Example 2.2 Applying Convolution theorem, verify that

$$\sin t \star \cos t = \int_0^t \sin u \cos(t-u)du = \frac{1}{2} t \sin t$$

Solution: Let $f(t) = \sin t$, $g(t) = \cos t$. Then $L[\sin t] = \frac{1}{s^2+1} = F(s)$, $L[g(t)] = \frac{s}{s^2+1} = G(s)$.
By convolution theorem, $L^{-1}\Big[F(s)\cdot G(s)\Big] = f \star g = \int_0^t f(u)g(t-u)du$.

$$\begin{aligned} \int_0^t \sin u \cos(t-u)du &= L^{-1}\Big[\frac{s}{(s^2+1)^2}\Big] \\ &= tL^{-1}\Big[\int_s^\infty \frac{u\,du}{(u^2+1)^2}\Big] \qquad (2.1) \end{aligned}$$

$$\begin{aligned} \text{Now } \int_s^\infty \frac{u}{(u^2+1)^2}du &= \frac{1}{2}\int_s^\infty \frac{2u\,du}{(u^2+1)^2} \\ &= \frac{1}{2}\lim_{B\to\infty}\int_s^B \frac{2u\,du}{(u^2+1)^2} \\ &= \frac{1}{2}\lim_{B\to\infty}\Big[-\frac{1}{u^2+1}\Big]_s^B = \frac{1}{2}\frac{1}{s^2+1} \end{aligned}$$

$$\therefore L^{-1}\Big[\int_s^\infty \frac{u\,du}{(u^2+1)^2}\Big] = \frac{t}{2}L^{-1}\Big[\frac{1}{(s^2+1)^2}\Big] = \frac{1}{2}t\sin t \qquad (2.2)$$

From (2.1)-(2.2), we get, $\int_0^t \sin u \cos(t-u)du = \frac{1}{2} t \sin t$. Hence verified. (2.3)

Example 2.3 Evaluate by Convolution theorem, $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]$

Solution: Let $L^{-1}\left[\frac{1}{(s-2)}\right] = e^{2t} = f(t)$, $L^{-1}\left[\frac{1}{(s^2+1)}\right] = \sin t = g(t)$.

$$
\begin{aligned}
L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right] &= L^{-1}\left[\frac{1}{(s-2)} \cdot \frac{1}{s^2+1}\right] \\
&= f(t) \star g(t) \\
&= \int_0^t f(u)g(t-u)du = \int_0^t e^{2u} \sin(t-u)du \\
&= \int_0^t e^{2u}\left\{\sin t \cos u - \cos t \sin u\right\}du \\
&= \sin t \int_0^t e^{2u} \cos u du - \cos t \int_0^t e^{2u} \sin u du \\
&= \frac{1}{5}\left[e^{2t} - 2\sin t - \cos t\right]
\end{aligned}
$$

[**Note** $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\left(a \sin bx - b \cos bx\right) + c$ & $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}\left(a \cos bx + b \sin bx\right) + c$]

Example 2.4 Apply Convolution theorem to evaluate $L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\}$.

Solution: $L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\} = L^{-1}\left\{\frac{1}{(s^2+2s+5)} \cdot \frac{1}{(s^2+2s+5)}\right\} = L^{-1}\left\{F(s) \cdot F(s)\right\}$

where $F(s) = \dfrac{1}{(s^2+2s+5)}$

so $f(t) = L^{-1}\left\{\dfrac{1}{(s^2+2s+5)}\right\}$, [By applying the inverse of Laplace Transform]

$= L^{-1}\left\{\dfrac{1}{(s+1)^2+2^2}\right\} = e^{-t}L^{-1}\left\{\dfrac{1}{s^2+2^2}\right\}$, [by First shifting property]

$= e^{-t}\dfrac{\sin 2t}{2}$

By convolution theorem,

$$\begin{aligned}
L^{-1}\Big\{F(s)\cdot F(s)\Big\} &= \int_0^t f(u)\cdot f(t-u)\,du \\
\Rightarrow L^{-1}\Big\{\frac{1}{(s^2+s+5)^2}\Big\} &= \int_0^t \Big(e^{-u}\frac{\sin 2u}{2}\Big)\cdot\Big(e^{-(t-u)}\frac{\sin 2(t-u)}{2}\Big)\,du \\
&= \frac{e^{-t}}{4}\int_0^t \sin 2u\cdot\sin 2(t-u)\,du \\
&= \frac{e^{-t}}{4}\cdot\frac{1}{2}\int_0^t \Big[\cos(4u-2t)-\cos 2t\Big]\,du \\
&= \frac{e^{-t}}{8}\Big[\frac{\sin(4u-2t)}{4}-u\cos 2t\Big]_0^t \\
&= \frac{e^{-t}}{8}\Big[\frac{\sin(2t)}{4}-t\cos 2t-\frac{\sin(-2t)}{4}\Big] \\
&= \frac{e^{-t}}{8}\Big[\frac{\sin(2t)}{2}-t\cos 2t\Big]
\end{aligned}$$

Example 2.5 Prove that the n-fold repeated integral

$$\int_0^t\int_0^t\cdots\int_0^t f(\xi)(d\xi)^n = \int_0^t \frac{f(x)(t-x)^{n-1}}{(n-1)!}dx.$$

Solution:

$$\begin{aligned}
&\because L\Big\{\int_0^t f(\xi)d\xi\Big\} = \frac{F(s)}{s} \\
&\therefore L\Big\{\int_0^t\Big\{\int_0^t f(\xi)d\xi\Big\}d\xi\Big\} = \frac{L\Big\{\int_0^t f(\xi)d\xi\Big\}}{s} = \frac{F(s)}{s^2} \\
&\qquad\cdots\qquad\cdots\qquad\cdots \\
&\text{and } L\Big\{\int_0^t\int_0^t\cdots\int_0^t f(\xi)(d\xi)^n\Big\} = \frac{F(s)}{s^n} \qquad (2.4)
\end{aligned}$$

Now, $L^{-1}\Big\{F(s)\Big\} = f(t)$ and $L^{-1}\Big\{s^{-n}\Big\} = \frac{t^{n-1}}{(n-1)!} = g(t)$ (say). Hence by convolution theorem,

$$\begin{aligned}
L^{-1}\Big\{F(s)\cdot G(s)\Big\} &= f(t)\star g(t) = \int_0^t f(x)g(t-x)dx \\
\therefore L^{-1}\Big\{\frac{F(s)}{s^n}\Big\} &= \int_0^t \frac{f(x)(t-x)^{n-1}}{(n-1)!}dx \qquad (2.5)
\end{aligned}$$

From the equations (2.4) and (2.5), we have $\int_0^t\int_0^t\cdots\int_0^t f(\xi)(d\xi)^n = \int_0^t \frac{f(x)(t-x)^{n-1}}{(n-1)!}dx$.

Example 2.6 For the Beta function B defined by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, m>0, n>0$$

prove that $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Proof: Let $M(t) = \int_0^t x^{m-1}(t-x)^{n-1}dx$ and using the convolution rule we have

$$\begin{aligned} L\Big\{M(t)\Big\} &= L\Big\{\int_0^t x^{m-1}(t-x)^{n-1}dx\Big\} \\ &= L\Big\{t^{m-1}\Big\}\cdot L\Big\{t^{n-1}\Big\} = \frac{\Gamma(m)}{s^m}\cdot\frac{\Gamma(n)}{s^n}, m>0, n>0 \\ &= \frac{\Gamma(m)\cdot\Gamma(n)}{s^{m+n}} \end{aligned}$$

Taking the inverse transform, we get $M(t) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}t^{m+n-1} = \int_0^t x^{m-1}(t-x)^{n-1}dx$. Hence putting $t=1$, we get $M(1) = B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$.

Solution of differential equations by Laplace Transform:

Example 2.7 Solve the equation using the Laplace transform method

$$y'' + 4y' + 8y = \cos 2t \tag{2.6}$$

Given that $y = 2$ and $y' = 1$ when $t = 0$.

Solution: Applying Laplace transform in both sides with respect to t in the equation (2.6), we obtain $\{s^2Y(s) - sy(0) - y'(0)\} + 4\{sY(s) - y(0)\} + 8Y(s) = \frac{s}{s^2+4}$. Using the initial conditions, we get, $s^2Y(s) - 2s - 1 + 4sY(s) - 8 + 8Y(s) = \frac{s}{s^2+4}$ or $(s^2+4s+8)Y(s) = \frac{s}{s^2+4} + 2s + 9$.

$$\begin{aligned} \text{Therefore } Y(s) &= \frac{1}{20}\times\frac{s}{s^2+4} + \frac{1}{5}\times\frac{1}{s^2+4} - \frac{1}{20}\times\frac{(s+2)-2}{(s+2)^2+2^2} \\ &- \frac{2}{5}\times\frac{1}{(s+2)^2+2^2} + \frac{2(s+2)-4}{(s+2)^2+2^2} + \frac{9}{(s+2)^2+2^2} \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} \text{Therefore } y(t) &= \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t - \frac{1}{20}e^{-2t}\cos 2t + \frac{1}{20}e^{-2t}\sin 2t \\ &- \frac{2}{10}e^{-2t}\sin 2t + 2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t + \frac{9}{2}e^{-2t}\sin 2t \end{aligned}$$

On simplification, we get $y(t) = \frac{e^{-2t}}{20}(39\cos 2t + 47\sin 2t)) + \frac{1}{20}(\cos 2t + 2\sin 2t)$.

Example 2.8 Solve the equation using the Laplace transform method

$$t\,y'' + y' + ty = 0 \tag{2.7}$$

Given that $y = 1$ and $y' = 0$ when $t = 0$.

Solution: Applying Laplace transform in both sides with respect to t in the equation (2.7), we obtain $-\frac{d}{ds}\{s^2Y(s) - sy(0) - y'(0)\} + \{sY(s) - y(0)\} - \frac{d}{ds}\{Y(s)\} = 0$. Using the initial conditions, we get, $-\frac{d}{ds}\{s^2Y(s) - s\} + \{sY(s) - 1\} - \frac{dY(s)}{ds} = 0$ or $(s^2+1)\frac{dY(s)}{ds} + sY(s) = 0$ which is a first order ODE. Integrating, we get $ln\,Y(s) + \frac{1}{2}ln(s^2+1) = ln\,c$ or $Y(s) = \frac{c}{\sqrt{s^2+1}}$. Taking the inverse

Laplace transform, we obtain $y(t) = c\,J_0(t)$, where $J_0(t)$ is a Bessel function of order zero. Since $y(0) = 1 = c\,J_0(0) = c$, the required solution is $y(t) = J_0(t)$.

Example 2.9 Solve the simultaneous equations using the Laplace transform method

$$\frac{dx}{dt} - y = e^t, \quad \frac{dy}{dt} + x = \sin\, t \tag{2.8}$$

Given that $x(0) = 1$ and $y(0) = 0$.

Solution: Applying Laplace transform in both equations and using the notation $Y = L[y; s]$,, we get, $sX - x(0) - Y = \frac{1}{s-1}$, $sY - y(0) + X = \frac{1}{s^2+1}$. Using the given initial conditions, the above equations reduce to

$$sX - Y = \frac{s}{s-1} \tag{2.9}$$

$$X + sY = \frac{1}{s^2+1} \tag{2.10}$$

Solving equations (2.9) and (2.10), we get

$$\begin{aligned} X &= \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2} \\ &= \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) + \frac{1}{(s^2+1)^2} \end{aligned} \tag{2.11}$$

$$\begin{aligned} Y &= \frac{s^3}{(s-1)(s^2+1)} + \frac{s}{(s^2+1)^2} - \frac{s}{s-1} \\ &= \frac{s}{(s^2+1)^2} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \end{aligned} \tag{2.12}$$

Again, taking the inverse Laplace transform of (2.11), we obtain

$$x(t) = \frac{1}{2}[e^t + \cos\, t + \sin\, t + (\sin\, t - t\cos\, t)] \tag{2.13}$$

Again, taking the inverse Laplace transform of (2.12), we obtain

$$y(t) = \frac{1}{2}[t\sin\, t - e^t + \cos\, t - \sin\, t] \tag{2.14}$$

Equations (2.13) and (2.14) constitute the solution of the given system.

Example 2.10 Solve one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{2.15}$$

Subject to the

$$\text{ICS:} \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, x > 0 \tag{2.16}$$

$$\text{BCS:} \quad u(0,t) = F(t), \quad u(x,t) \to 0, \text{ as } x \to \infty,\ t \geq 0 \tag{2.17}$$

Solution: Applying Laplace transform in both sides with respect to t in the equation (2.15), we get

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = c^2 L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow c^2\frac{d^2U(x,s)}{dx^2} = s^2U(x,s) - su(x,0) - u_t(x,0)[\text{Where } L\{u(x,t)\} = U(x,s)]$$

$$\Rightarrow \frac{d^2U(x,s)}{dx^2} = \frac{s^2}{c^2}U(x,s) \text{ [From the given conditions]} \quad (2.18)$$

Solving the equation (2.18),we get $U(x,s) = Ae^{\frac{s}{c}x} + Be^{-\frac{s}{c}x}$ (2.19)

Since $u(0,t) = F(t), u(x,t) \to 0$ as $x \to \infty,\ t \geq 0$. Their Laplace Transformations are

$U(0,s) = f(s)$ [where $L\{F(t)\} = f(s)$] (2.20)

$U(x,s) \to 0$ as $x \to \infty$ (2.21)

From the equations (2.19) and (2.20), we get $A + B = f(s)$ (2.22)

From the equations (2.19) and (2.21), we get $A = 0$ (2.23)

Solving (2.22) and (2.23), we get $B = f(s)$ (2.24)

Now from (2.19), we get $U(x,s) = f(s)e^{-\frac{s}{c}x}$ (2.25)

Taking inverse Laplace Transformation, we get the general solution as

$$\begin{aligned} u(x,t) &= L^{-1}\Big\{U(x,s)\Big\} = L^{-1}\Big\{f(s)e^{-\frac{s}{c}x}\Big\} = \begin{cases} F(t-\frac{x}{c}), & t \geq \frac{x}{c} \geq 0 \\ 0, & t < \frac{x}{c} \end{cases} \\ &= F(t-\frac{x}{c})H(t-\frac{x}{c}),\ \text{where}\ H(t-\frac{x}{c}) = \begin{cases} 1, & t \geq \frac{x}{c} \geq 0 \\ 0, & t < \frac{x}{c} \end{cases} \end{aligned}$$

Example 2.11 Solve one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad (2.26)$$

Subject to the

$$\text{BCS:} \quad u(0,t) = 0, \qquad u(1,t) = 0, t > 0 \quad (2.27)$$

$$\text{ICS:} \quad u(x,0) = \sin \pi x, \qquad u_t(x,0) = -\sin \pi x, 0 < x < 1 \quad (2.28)$$

Solution: Applying Laplace transform in both sides in the equation (2.26), we get

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = L\left\{\frac{\partial^2 u}{\partial x^2}\right\} \Rightarrow \frac{d^2U(x,s)}{dx^2} = s^2U(x,s) - su(x,0) - u_t(x,0)$$

$$\Rightarrow \frac{d^2U(x,s)}{dx^2} = s^2U(x,s) - s\sin\pi x + \sin\pi x = s^2U(x,s) + (1-s)\sin\pi x \quad (2.29)$$

Solving the equation (2.29), we get $U(x,s) = Ae^{sx} + Be^{-sx} - \dfrac{(1-s)\sin\pi x}{\pi^2 + s^2}$ (2.30)

Since $u(0,t) = 0, u(1,t) = 0, t > 0$. Their LT are $U(0,s) = 0$ (2.31)

and $U(1,s) = 0$ (2.32)

From the equations (2.30) and (2.31), we get $A + B = 0$ (2.33)

From the equations (2.30) and (2.32), we get $Ae^s + Be^{-s} = 0$ (2.34)

Solving (2.33) and (2.34), we get, $A = B = 0$ and so, $U(x,s) = -\frac{(1-s)\sin\pi x}{\pi^2+s^2}$.
Taking inverse Laplace Transformation, we get the general solution as

$$\begin{aligned} u(x,t) &= L^{-1}\Big\{U(x,s)\Big\} = \sin\pi x\Big[L^{-1}\Big\{\frac{s}{\pi^2+s^2}\Big\} - L^{-1}\Big\{\frac{1}{\pi^2+s^2}\Big\}\Big] \\ \Rightarrow u(x,t) &= \sin\pi x\Big[\cos\pi t - \frac{\sin\pi t}{\pi}\Big] \end{aligned}$$

Example 2.12 Solve one dimensional Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{2.35}$$

Subject to the

$$\text{BCS:} \quad u(0,t) = 1, u(1,t) = 1, \ t > 0 \tag{2.36}$$

$$\text{IC:} \quad u(x,0) = 1 + \sin\pi x, \ 0 < x < 1 \tag{2.37}$$

Solution: Applying Laplace transform in both sides in the equation (2.35), we get

$$L\Big\{\frac{\partial u}{\partial t}\Big\} = L\Big\{\frac{\partial^2 u}{\partial x^2}\Big\} \Rightarrow \frac{d^2U(x,s)}{dx^2} = sU(x,s) - u(x,0)$$

$$\Rightarrow \frac{d^2U(x,s)}{dx^2} = sU(x,s) - 1 - \sin\pi x$$

$$\Rightarrow \frac{d^2U(x,s)}{dx^2} - sU(x,s) = -(1+\sin\pi x) \tag{2.38}$$

Solving the equation (2.38), we get $U(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin\pi x}{\pi^2+s}$ (2.39)

Since $u(0,t) = 1, u(1,t) = 1, t > 0$. Their LT are $U(0,s) = \frac{1}{s}$ (2.40)

and $U(1,s) = \frac{1}{s}$ (2.41)

From the equations (2.39) and (2.40), we get $A + B + \frac{1}{s} = \frac{1}{s}$ (2.42)

From the equations (2.39) and (2.41), we get $Ae^{\sqrt{s}} + Be^{-\sqrt{s}} + \frac{1}{s} = \frac{1}{s}$ (2.43)

Solving (2.42) and (2.43), we get $A = B = 0$ and so $U(x,s) = \frac{1}{s} + \frac{\sin\pi x}{\pi^2+s}$ (2.44)

Taking inverse Laplace Transformation, we get the general solution as

$$\begin{aligned} u(x,t) &= L^{-1}\Big\{U(x,s)\Big\} = L^{-1}\Big\{\frac{1}{s}\Big\} + L^{-1}\Big\{\frac{\sin\pi x}{\pi^2+s}\Big\} \\ \Rightarrow u(x,t) &= 1 + \sin\pi x e^{-\pi^2 t} \end{aligned}$$

Example 2.13 Solve one dimensional Heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \tag{2.45}$$

Subject to the

$$\text{BCS:} \quad u(0,t) = 0, \qquad u(5,t) = 0,\ t > 0 \tag{2.46}$$

$$\text{IC:} \quad u(x,0) = 10\sin 4\pi x,\ 0 < x < 5 \tag{2.47}$$

Solution: Applying Laplace transform in both sides in the equation (2.45), we get

$$L\left\{\frac{\partial u}{\partial t}\right\} = 2L\left\{\frac{\partial^2 u}{\partial x^2}\right\} \Rightarrow 2\frac{d^2U(x,s)}{dx^2} = sU(x,s) - u(x,0)$$

$$\Rightarrow 2\frac{d^2U(x,s)}{dx^2} = sU(x,s) - 10\sin 4\pi x$$

$$\Rightarrow \frac{d^2U(x,s)}{dx^2} - \frac{s}{2}U(x,s) = -5\sin 4\pi x \tag{2.48}$$

Solving the equation (2.48), we get $U(x,s) = Ae^{\sqrt{\frac{s}{2}}x} + Be^{-\sqrt{\frac{s}{2}}x} + \dfrac{(10\sin 4\pi x)}{32\pi^2 + s}$ (2.49)

Since $u(0,t) = 0, u(5,t) = 0, t > 0$. Their LT are $U(0,s) = 0$ (2.50)

and $U(5,s) = 0$ (2.51)

From the equations (2.49) and (2.50), we get $A + B = 0$ (2.52)

From the equations (2.49) and (2.51), we get $Ae^{\sqrt{\frac{s}{2}}5} + Be^{-\sqrt{\frac{s}{2}}5} = 0$ (2.53)

Solving (2.52) and (2.53), we get $A = B = 0$ (2.54)

So from (2.49), we get $U(x,s) = \dfrac{10\sin 4\pi x}{32\pi^2 + s}$ (2.55)

Taking inverse Laplace Transformation, we get the general solution as

$$u(x,t) = L^{-1}\left\{U(x,s)\right\} = (10\sin 4\pi x)L^{-1}\left\{\frac{1}{32\pi^2 + s}\right\}$$

$$\Rightarrow u(x,t) = 10\sin 4\pi x e^{-32\pi^2 t}.$$

Example 2.14 Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 1 \tag{2.56}$$

with initial conditions $f(0) = f'(0) = 0$.

Solution: First, take the Laplace transform of the equation. Since $f'(0) = f(0) = 0$, if $L(f) = F(s)$ then $L(f') = sF(s)$ and $L(f'') = s^2F(s)$. Thus, the subsidiary equation is

$$s^2F - 4sF + 3F = \frac{1}{s} \text{ and so } (s^2 - 4s + 3)F = \frac{1}{s}$$

$$F = \frac{1}{s}\frac{1}{s^2 - 4s + 3} \text{ and since } s^2 - 4s + 3 = (s-3)(s-1), \text{ this gives } F = \frac{1}{s(s-3)(s-1)}$$

Before we can invert this, we need to do a partial fraction expansion.

$$\frac{1}{s(s-3)(s-1)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1} \Rightarrow A(s-3)(s-1) + Bs(s-1) + Cs(s-3) = 1$$

So substituting in $s = 0$ we get $A = 1/3$, $s = 3$ gives $B = 1/6$ and $s = 1$ gives $C = -1/2$. Hence

$$F = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)}$$

and taking inverse LT we get,

$$f(t) = \frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^{t}$$

Example 2.15 Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 2e^t \tag{2.57}$$

with initial conditions $f(0) = f'(0) = 0$.

Solution: This time we have $L(2e^t) = 2/(s-1)$ on the right hand side. This means that the subsidiary equation is

$$(s^2 - 4s + 3)F = \frac{2}{s-1} \qquad \text{so } F = \frac{2}{(s-1)^2(s-3)} \tag{2.58}$$

We need to do partial fractions again, but this is one of those cases with a repeated root:

$$\frac{1}{(s-1)^2(s-3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-3} \Rightarrow A(s-1)(s-3) + B(s-3) + C(s-1)^2 = 1 \tag{2.59}$$

So $s = 1$ gives $B = -1/2$ and $s = 3$ gives $C = 1/4$. No value of s gives A on its own, so wee try $s = 2$:

$$1 = -A + \frac{1}{2} + \frac{1}{4}$$

which means that $A = -1/4$. Hence

$$F = -\frac{1}{2(s-1)} - \frac{1}{(s-1)^2} + \frac{1}{2(s-3)}$$

and taking inverse LT, we get,

$$f(t) = -\frac{1}{2}e^t - te^t + \frac{1}{2}e^{3t}$$

Example 2.16 Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 0 \tag{2.60}$$

with initial conditions $f(0) = 1$ and $f'(0) = 1$.

Solution: In this example there are non-zero boundary conditions. Since

$$L(f') = sF - f(0) \tag{2.61}$$
$$L(f'') = s^2F - sf(0) - f'(0) \tag{2.62}$$

the subsidiary equation in this case is

$$s^2F - s - 1 - 4sF + 4 + 3F = 0 \qquad \text{so } (s^2 - 4s + 3)F = s - 3.$$
$$\text{Hence } F = \frac{1}{s-1} \qquad \text{and taking LT, we get, } f(t) = e^t$$

Example 2.17 Using the Laplace transform solve the differential equation

$$y'' - 2ay' + a^2y = 0 \tag{2.63}$$

with initial conditions $y'(0) = 1$ and $y(0) = 0$. a is some real constant.

Solution: Taking the Laplace transform we get

$$s^2Y - 1 - 2aY + a^2Y = 0 \Rightarrow Y = \frac{1}{(s-a)^2} \tag{2.64}$$

$$\text{Taking Inverse Laplace tranformation, we get, } \; y = te^{at} \tag{2.65}$$

Example 2.18 Using the Laplace transform solve the differential equation

$$f'' + f' - 6f = e^{-3t} \tag{2.66}$$

with initial conditions $f(0) = f'(0) = 0$.

Solution: So, the subsidiary equation is $s^2F + sF - 6F = \frac{1}{s+3} \Rightarrow F(s) = \frac{1}{(s+3)^2(s-2)}$. We do partial fractions

$$\frac{1}{(s+3)^2(s-2)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s-2} \Rightarrow A(s+3)(s-2) + B(s-2) + C(s+3)^2 = 1 \tag{2.67}$$

Taking $s = -3$ gives $B = -1/5$ and $s = 2$ gives $C = 1/25$. Putting in $s = 1$ we find

$$4A + \frac{1}{5} + \frac{16}{25} = 1 \text{ and so } A = -1/25. \tag{2.68}$$

$$\text{Putting the values of } A, B, C, \text{ we get } \; F(s) = -\frac{1}{25(s+3)} - \frac{1}{5(s+3)^2} + \frac{1}{25(s-2)} \tag{2.69}$$

$$\text{Taking Inverse Laplace transformation, we get, } \; f(t) = -\frac{1}{25}e^{-3t} - \frac{t}{5}e^{-3t} + \frac{1}{25}e^{2t} \tag{2.70}$$

Example 2.19 Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = 0 \tag{2.71}$$

with initial conditions $f(0) = 0$ and $f'(0) = 1$.

Solution: Taking the Laplace transform of the equation we get,

$$s^2F - 1 + 6sF + 13F = 0 \Rightarrow F = \frac{1}{s^2 + 6s + 13}. \tag{2.72}$$

Now, using minus b plus or minus the square root of b squared minus four a c all over two a, we get

$$s^2 + 6s + 13 = 0 \Rightarrow s = \frac{-6 \pm \sqrt{36 - 52}}{2} = -3 \pm 2i \tag{2.73}$$

$$\text{which means } s^2 + 6s + 13 = (s + 3 - 2i)(s + 3 + 2i) \tag{2.74}$$

Next, we do the partial fraction expansion,

$$\frac{1}{s^2 + 6s + 13} = \frac{A}{s + 3 - 2i} + \frac{B}{s + 3 + 2i} \Rightarrow A(s + 3 + 2i) + B(s + 3 - 2i) = 1. \tag{2.75}$$

Therefore we choose $s = -3+2i$ to get $A = \frac{1}{4i} = -\frac{i}{4}$ and $s = -3-2i$ to get $B = -\frac{1}{4i} = \frac{i}{4}$ and so

$$F = -\frac{i}{4}\frac{1}{s+3-2i} + \frac{i}{4}\frac{1}{s+3+2i}. \tag{2.76}$$

Then by taking inverse Laplace transform, we get

$$f(t) = -\frac{i}{4}e^{-(3-2i)t} + \frac{i}{4}e^{-(3+2i)t} = \frac{i}{4}e^{-3t}(e^{-2it} - e^{2it}) = \frac{1}{2}e^{-3t}\sin 2t \tag{2.77}$$

Example 2.20 Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = e^t \tag{2.78}$$

with initial conditions $f(0) = 0$ and $f'(0) = 0$.

Solution: Taking the Laplace transform of the equation gives

$$s^2F + 6sF + 13F = \frac{1}{s-1} \Rightarrow F = \frac{1}{(s-1)(s+3+2i)(s+3-2i)}. \tag{2.79}$$

$$\text{We write, } \frac{1}{(s-1)(s+3+2i)(s+3-2i)} = \frac{A}{s+3-2i} + \frac{B}{s+3+2i} + \frac{C}{s-1}$$
$$\Rightarrow A(s-1)(s+3+2i) + B(s-1)(s+3-2i) + C(s+3-2i)(s+3+2i) = 1.$$

Putting $s = -3+2i$ we get $A(-4+2i)(4i) = A(-8-16i) = 1$ so $A = -\frac{1}{8+16i} = -\frac{1}{8+16i}\frac{8-16i}{8-16i} = -\frac{1+2i}{40}$. In the same way, $s = -3-2i$ leads to $B = -\frac{1-2i}{40}$ and finally, $s = 1$ gives $C = \frac{1}{20}$. Putting all this together we get

$$F(s) = -\frac{1+2i}{40}\frac{1}{s+3-2i} - \frac{1-2i}{40}\frac{1}{s+3+2i} + \frac{1}{20}\frac{1}{s-1} \tag{2.80}$$

$$\begin{aligned} \text{and so } f(t) &= -\frac{1+2i}{40}e^{-(3-2i)t} - \frac{1-2i}{40}e^{-(3+2i)t} + \frac{1}{20}e^t \\ &= -\frac{1}{40}e^{-3t}\left[(1+2i)e^{2it} + (1-2i)e^{-2it}\right] + \frac{1}{20}e^t \end{aligned} \tag{2.81}$$

We then substitute in $e^{2it} = \cos 2t + i\sin 2t$, $e^{-2it} = \cos 2t - i\sin 2t$ to end up with $f(t) = \frac{1}{20}e^{-3t}[2\sin 2t - \cos 2t] + \frac{1}{20}e^t$.

Example 2.21 Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \begin{cases} 1, & 0 \le t < c \\ 0, & t \ge c \end{cases} \tag{2.82}$$

subject to the initial conditions $f(0) = f'(0) = 0$.

Solution: Taking Laplace transforms of both sides and using the tables for the Laplace transform of the right hand side function, leads to

$$\begin{aligned} (s^2+2s-3)F &= \frac{1-e^{-cs}}{s} \Rightarrow F = \frac{1-e^{-cs}}{s(s^2+2s-3)} \\ &= (1-e^{-cs})\frac{1}{s(s-1)(s+3)} = (1-e^{-cs})\left(\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}\right) \end{aligned} \tag{2.83}$$

Concentrating on the partial fractions part, we have

$$\frac{1}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3} \Rightarrow A(s-1)(s+3) + Bs(s+3) + Cs(s-1) = 1$$

$\underline{s = 0:}$ $\quad -3A = 1 \quad \Rightarrow A = -\frac{1}{3}$

$\underline{s = 1:}$ $\quad 0 + 4B + 0 = 1 \quad \Rightarrow B = \frac{1}{4}$

$\underline{s = -3:}$ $\quad 0 + 012C = 1 \quad \Rightarrow C = \frac{1}{12}$

Hence we have $F = (1 - e^{-cs})\left(-\frac{1}{3}\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{12}\frac{1}{s+3}\right)$

From the tables, we know that $L\left(-\frac{1}{3} + \frac{1}{4}e^{t} - \frac{1}{12}e^{-3t}\right) = -\frac{1}{3}\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{12}\frac{1}{s+3}$

and then using the second shift theorem

$$f(t) = -\frac{1}{3} + \frac{1}{4}e^{t} + \frac{1}{12}e^{-3t} - H_c(t)\left(-\frac{1}{3} + \frac{1}{4}e^{(t-c)} + \frac{1}{12}e^{-3(t-c)}\right) \tag{2.84}$$

Example 2.22 Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \tag{2.85}$$

subject to the initial conditions $f(0) = 0$ and $f'(0) = 0$.

Solution: Remember the definition of the Heaviside function:

$$H_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \tag{2.86}$$

so the Heaviside function is zero until a and then it is one. The right hand side is zero until $t = 1$ and then it is one until $t = 2$ and then it is zero again. Consider $H_1(t) - H_2(t)$, this is zero until you reach $t = 1$, then the first Heaviside function switches on, the other one remains zero. Things stay like this until you reach $t = 2$, then the second Heaviside function switches on as well and you get $1 - 1 = 0$. Thus

$$H_1(t) - H_2(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \tag{2.87}$$

Now, using

$$L(H_a(t)) = \frac{e^{-as}}{s} \tag{2.88}$$

we take the Laplace transform of the differential equation:

$$s^2F + 2sF - 3F = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \Rightarrow F(s) = \frac{1}{s(s-1)(s+3)}\left(e^{-s} - e^{-2s}\right)$$

Now, we have that $\frac{1}{s(s-1)(s+3)} = -\frac{1}{3s} + \frac{1}{4(s-1)} + \frac{1}{12(s+3)}$

and we know that $L\left(-\frac{1}{3} + \frac{1}{4}e^{t} + \frac{1}{12}e^{-3t}\right) = -\frac{1}{3} + \frac{1}{4(s-1)} + \frac{1}{12(s+3)}$

In other word, if it wasn't for the exponentials we'd know the little f. However, we know from the second shift theorem that the affect of the exponential e^{-as} is to change t to $t-a$ and to introduce an overall factor of $H_a(t)$. Thus

$$f(t) = H_1(t)\left(-\frac{1}{3}+\frac{1}{4}e^{t-1}+\frac{1}{12}e^{-3t+3}\right) - H_2(t)\left(-\frac{1}{3}+\frac{1}{4}e^{t-2}+\frac{1}{12}e^{-3t+6}\right).$$

Example 2.23 Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \delta(t-1) \tag{2.89}$$

subject to the initial conditions $f(0) = 0$ and $f'(0) = 1$.

Solution: We take the Laplace transform using

$$L(\delta(t-a)) = e^{-as} \Rightarrow (s^2+2s-3)F - 1 = e^{-s} \tag{2.90}$$

Now, if we do partial fractions on $1/(s^2+2s-3)$ we get

$$\frac{1}{s^2+2s-3} = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} \tag{2.91}$$

$$\text{Hence } F = \left(-\frac{1}{4(s+3)} + \frac{1}{4(s-1)}\right)(1+e^{-s}) \tag{2.92}$$

$$\text{Since } L\left(-\frac{1}{4}e^{-3t}+\frac{1}{4}e^{t}\right) = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} \tag{2.93}$$

then, by the second shift theorem we have

$$f = \left(-\frac{1}{4}e^{-3t}+\frac{1}{4}e^{t}\right) + H_1(t)\left(-\frac{1}{4}e^{-3t+3}+\frac{1}{4}e^{t-1}\right) \tag{2.94}$$

Example 2.24 Consider the Laplace transform to solve

$$f'' + 2f' + 5f = 1 \tag{2.95}$$

with initial conditions $f(0) = f'(0) = 0$.

Solution: Taking the Laplace transform of each side we get

$$(s^2+2s+5)F = \frac{1}{s} \Rightarrow F = \frac{1}{s(s^2+2s+5)} \tag{2.96}$$

We now need to factorize s^2+2s+5. Solving using the formula gives

$$s = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i \tag{2.97}$$

$$\text{so } s^2+2s+5 = (s+1+2i)(s+1-2i) \tag{2.98}$$

Next, partial fractions. This part is no different from the examples without complex numbers, but it is trickier.

$$\frac{1}{s(s+1-2i)(s+1+2i)} = \frac{A}{s} + \frac{B}{s+1-2i} + \frac{C}{s+1+2i} \tag{2.99}$$

Multiplying across by the denominator gives

$$A(s+1-2i)(s+1+2i)+Bs(s+1+2i)+Cs(s+1-2i)=1 \tag{2.100}$$

Choosing $s=0$ gives $1=5A$ and hence $A=1/5$. Next, $s=-1+2i$ gives $1=B(-1+2i)(4i)=-8-4i$ so $B=-\frac{1}{8+4i}$. Finally, $s=-1-2i$ gives $C(-1-2i)(-4i)=-8+4i=1$ giving $C=-\frac{1}{8-4i}$. Putting all this together we get

$$F=\frac{1}{5s}-\frac{1}{8+4i}\frac{1}{s+1-2i}-\frac{1}{8-4i}\frac{1}{s+1+2i} \tag{2.101}$$

and, using the tables, this gives

$$f=\frac{1}{5}-\frac{1}{8+4i}e^{-(1-2i)t}-\frac{1}{8-4i}e^{-(1+2i)t} \tag{2.102}$$

Although this does tell us what f is, it does it in a complicated way. For a start, this makes it look like f is complex, when we know that f satisfies a real differential equation and should be real. To rewrite this in a real form we need to do two things, we change the fractions with complex numbers on the bottom to fractions with complex numbers on the top and we expand the exponentials using the formula

$$\begin{aligned} e^{i\theta} &= \cos\theta+i\sin\theta \\ e^{-i\theta} &= \cos\theta-i\sin\theta \end{aligned} \tag{2.103}$$

Note that the second of these formulas follows from the first using

$$\begin{aligned} \cos(-\theta) &= \cos\theta \\ sin(-\theta) &= -\sin\theta \end{aligned} \tag{2.104}$$

First the fractions, remember there is a standard method for dividing by a complex number: you multiply above and below by the conjugate. Hence

$$\frac{1}{8+4i}=\frac{1}{8+4i}\frac{8-4i}{8-4i} \tag{2.105}$$

which makes sense because the second fraction is equal to one. Now

$$(8+4i)(8-4i)=8^2-(4i)^2=64+16=80 \tag{2.106}$$

$$\text{and so }\frac{1}{8+4i}=\frac{8-4i}{80}=\frac{2-i}{20} \tag{2.107}$$

You can do the same with the other complex fraction, it is quicker just to note it is the same as the one we just did except the sign of i is different, so,

$$\frac{1}{8+4i}=\frac{8+4i}{80}=\frac{2+i}{20}$$

$$\text{Now we have } f(t)=\frac{1}{5}-\frac{2-i}{20}e^{-(1-2i)t}-\frac{2+i}{20}e^{-(1+2i)t}$$

$$\text{and so } f(t)=\frac{1}{5}-\frac{1}{20}\left[(2-i)e^{2it}-(2+i)e^{-2it}\right]e^{-t}$$

$$\text{i.e. } f(t)=\frac{1}{5}-\frac{1}{20}\left[(2-i)(\cos 2t+i\sin 2t)-(2+i)(\cos 2t+i\sin 2t)\right]e^{-t}$$

$$\text{or } f(t)=\frac{1}{5}-\frac{1}{10}(2\cos 2t+\sin 2t)e^{-t}$$

Example 2.25 Use the Laplace transform to solve the differential equation:

$$\frac{d^2 f}{dt^2} + 2\frac{df}{dt} + 2f = 1 \tag{2.108}$$

with initial conditions $f(0) = \frac{df}{dt}(0) = 0$.

Solution: Taking the Laplace transform of both sides we get.

$$(s^2 + 2s + 2)F = \frac{1}{s} \Rightarrow F(s) = \frac{1}{s(s^2 + 2s + 2)} \tag{2.109}$$

and we can factor the denominator using the roots

$$s = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm \sqrt{-1} = -1 \pm i$$

of the quadratic. So complex partial fractions take the form

$$\begin{aligned}
F &= \frac{1}{s(s-(-1+i))(s-(-1-i))} = \frac{A}{s} + \frac{B}{s+1-i} + \frac{C}{s+1+i} \\
&= A(s+1-i)(s+1+i) + Bs(s+1+i) + Cs(s+1-i) = 1 \\
&= A(s^2+2s+2) + Bs(s+1+i) + Cs(s+1-i) \\
\text{Putting} \quad & \underline{s = 0:} \qquad 2A = 1 \qquad \Rightarrow A = \frac{1}{2} \\
\text{Putting} \quad & \underline{s = -1+i:} \qquad 0 + B(-1+i)(2i) + 0 = 2B(-i-1) = -2(1+i)B = 1 \qquad (2.110) \\
\Rightarrow \quad & B = \frac{1}{-2(1+i)} = -\frac{1-i}{2(1+i)(1-i)} \\
&= -\frac{1-i}{4} = -\frac{1}{4} + \frac{i}{4} \\
\text{Putting} \quad & \underline{s = -1-i:} \qquad 0 + 0 + C(-1-i)(-2i) + 0 = 2C(i-1) = 2(-1+i)C = 1 \\
C &= \frac{1}{2(-1+i)} = \frac{-1-i}{2(-1+i)(-1-i)} \\
&= -\frac{1+i}{4} = -\frac{1}{4} - \frac{i}{4} \\
&= \overline{B} \\
\therefore F(s) &= \frac{1}{2}\frac{1}{s} + \left(-\frac{1}{4} + \frac{i}{4}\right)\frac{1}{s+1-i} + \left(-\frac{1}{4} - \frac{i}{4}\right)\frac{1}{s+1+i} \qquad (2.111)
\end{aligned}$$

Using inverse Laplace transform, we have

$$\begin{aligned}
f(t) &= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right)e^{(-1+i)t} + \left(-\frac{1}{4} - \frac{i}{4}\right)e^{(-1-i)t} \\
&= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right)e^{-t}e^{-it} + \left(-\frac{1}{4} - \frac{i}{4}\right)e^{-t}e^{-it} \\
&= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right)e^{-t}(\cos t + i\sin t) + \left(-\frac{1}{4} - \frac{i}{4}\right)e^{-t}(\cos t - i\sin t) \\
&= \frac{1}{2} + \frac{e^{-t}}{4}(-\cos t - \sin t - i\sin t + i\cos t) + \frac{e^{-t}}{4}(-\cos t - \sin t + i\sin t - i\cos t) \\
&= \frac{1}{2} - \frac{e^{-t}}{2}(\cos t + \sin t) \qquad (2.112)
\end{aligned}$$

Example 2.26 Consider the differential equation:

$$f'' + f' - 6f = \begin{cases} 0 & t < 3 \\ 2 & 3 \le t < 5 \\ 0 & 5 \le t \end{cases} \tag{2.113}$$

with initial conditions $f(0) = f'(0) = 0$.

Solution: The first thing to do is to rewrite the right hand side in terms of the Heaviside function. The key point is that the Heaviside function $H_a(t)$ is zero until you get to a and then it is one after that. Now the function on the right hand side is zero until we get to three and then it is two, so it behaves like $2H_3(t)$, however, at five it goes back down to zero, so we have to take away $2H_5(t)$, in short:

$$2H_3(t) - 2H_5(t) = \begin{cases} 0 & t < 3 \\ 2 & 3 \le t < 5 \\ 0 & 5 \le t \end{cases} \tag{2.114}$$

The first Heaviside function switches on at $t = 3$ and brings you up to two, the second switches on at five and brings you back down to zero.

Now, the differential equation is

$$f'' + f' - 6f = 2H_3(t) - 2H_5(t) \tag{2.115}$$

with $f(0) = f'(0) = 0$ and we take the Laplace transform of both sides:

$$s^2F + sF - 6F = \frac{2}{s}\left(e^{-3s} - e^{-5s}\right) \tag{2.116}$$

or

$$F = \frac{2}{s(s+3)(s-2)}\left(e^{-3s} - e^{-5s}\right) \tag{2.117}$$

Now, partial fractions has

$$\frac{2}{s(s+3)(s-2)} = -\frac{1}{3s} + \frac{2}{15}\frac{1}{s+3} + \frac{1}{5}\frac{1}{s-2} \tag{2.118}$$

Now,

$$-\frac{1}{3s} + \frac{2}{15}\frac{1}{s+3} + \frac{1}{5}\frac{1}{s-2} = L\left(-\frac{1}{3} + \frac{2}{15}e^{-3t} + \frac{1}{5}e^{2t}\right) \tag{2.119}$$

Now, in the expression for F this gets multiplied by various exponential factors, the effect of this is to delay the answer:

$$f = \left(-\frac{1}{3} + \frac{2}{15}e^{-3t+9} + \frac{1}{5}e^{2t-6}\right)H_3(t) + \left(-\frac{1}{3} + \frac{2}{15}e^{-3t+15} + \frac{1}{5}e^{2t-10}\right)H_5(t) \tag{2.120}$$

Now, here is a similar problem, but with a Dirac delta function:

$$f'' + f' - 6f = \delta(t-4) \tag{2.121}$$

with $f(0) = f'(0) = 0$. Using $L[\delta(t-a)] = e^{as}$ this gives

$$s^2F + sF - 6F = e^{4s} \tag{2.122}$$

or

$$F = \frac{e^{4s}}{(s+3)(s-2)} \tag{2.123}$$

By partial fractions we have

$$\frac{1}{(s+3)(s-2)} = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3} = L\left(\frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}\right) \tag{2.124}$$

so, the e^{4s} causes a delay of four and we have $f(t) = \left(\frac{1}{5}e^{2t-8} - \frac{1}{5}e^{-3t+12}\right)H_4(t)$.

Example 2.27 Consider

$$y'' - 4y' + 3y = 6t - 8 \tag{2.125}$$

with initial conditions $y(0) = y'(0) = 0$.

Solution: If we write $Y = L(y)$ the Laplace transform is

$$s^2Y - 4sY + 3Y = \frac{6}{s^2} - \frac{8}{s} \Rightarrow (s^2 - 4s + 3)Y = \frac{6}{s^2} - \frac{8}{s}$$
$$\Rightarrow Y = \frac{6}{s^2(s^2-4s+3)} - \frac{8}{s(s^2-4s+3)} \tag{2.126}$$

Now we have to put this into a form which allows us to take the inverse transform. The second term isn't so bad. Since $s^2 - 4s + 3 = (s-1)(s-3)$ we write

$$\frac{1}{s(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-3} \Rightarrow (s-1)(s-3)A + s(s-3)B + s(s-1)C = 1 \tag{2.127}$$

Thus, choosing $s = 0$ gives $A = 1/3$, $s = 1$ gives $B = -1/2$ and choosing $s = 3$ gives $C = 1/6$. Thus

$$\frac{1}{s(s^2-4s+3)} = \frac{1}{3s} - \frac{1}{2(s-1)} + \frac{1}{6(s-3)} \tag{2.128}$$

The other expansion is harder because it has a repeated root: in

$$\frac{1}{s^2(s-1)(s-3)} \tag{2.129}$$

the s factor appears as a square. To deal with this you have to include a $1/s$ term and a $1/s^2$ term in the partial fraction expansion.

$$\frac{1}{s^2(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-3}$$
$$\Rightarrow s(s-1)(s-3)A + (s-1)(s-3)B + s^2(s-3)C + s^2(s-1)D = 1$$

Now taking $s = 0$ gives $B = 1/3$, $s = 1$ gives $C = -1/2$ and $s = 3$ gives $D = 1/18$. There is no convenient choice of s that gives A on its own, so we just substitute in any other value, $s = 2$ say and by putting in the values of B, C and D we get

$$-2A - \frac{1}{3} + 2 + \frac{2}{9} = 1 \tag{2.130}$$

and hence $A = 4/9$. Thus
$$\frac{1}{s^2(s-1)(s-3)} = \frac{4}{9s} + \frac{1}{3s^2} - \frac{1}{2(s-1)} + \frac{1}{18(s-3)} \tag{2.131}$$

Now we can put everything together

$$\begin{aligned} Y &= 6\left(\frac{4}{9s}+\frac{1}{3s^2}-\frac{1}{2(s-1)}+\frac{1}{18(s-3)}\right) \\ &\quad -8\left(\frac{1}{3s}-\frac{1}{2(s-1)}+\frac{1}{6(s-3)}\right) \end{aligned} \tag{2.132}$$

and if we do the algebra we find $Y(s) = \dfrac{2}{s^2} + \dfrac{1}{s-1} - \dfrac{1}{s-3}$

Using inverse Laplace transform, we get, $y(t) = 2t + e^t - e^{3t}$

2.5 Multiple Choice Questions(MCQ)

1. The inverse transformation of $\frac{2s^2-4}{(s-3)(s^2-s-2)}$. **GATE(MA)-14**
   A) $(1+t)e^{-t} + \frac{7}{2}e^{-3t}$ B) $\frac{e^t}{3} + te^{-t} + 2t$ C) $\frac{7}{2}e^{3t} - \frac{e^{-t}}{6} - \frac{4}{3}e^{2t}$ D) $\frac{7}{2}e^{-3t} - \frac{e^{-t}}{6} - \frac{4}{3}e^{-2t}$
   Ans. (C)
2. If $F(s) = \tan^{-1}(s) + k$ is the Laplace transform of some function f on $t \geqslant 0$, then $k =$ **GATE(MA)-07**
   A) π B) $-\frac{\pi}{2}$ C) 0 D) $\frac{\pi}{2}$
   Ans. B)
   Hint. $L(f(t)) = \tan^{-1}(s) + k \Rightarrow f(t) = L^{-1}(\tan^{-1}(s) + k) = -\frac{\sin t}{t}$
   $\Rightarrow L\{-\frac{1}{t}\sin t\} = \tan^{-1} s - \frac{\pi}{2}$
3. Given two continuous time signals $x(t) = e^{-t}$ and $y(t) = e^{-2t}$, which exist for $t > 0$, the convolution $z(t) = x(t) \star y(t)$ is **GATE(EE)-11**
   A) $e^{-t} - e^{-2t}$ B) e^{-3t} C) e^{-t} D) $e^{-t} + e^{-2t}$
   Ans. (A) Taking Laplace transformation, we get

$$\begin{aligned} & L\Big\{z(t)\Big\} = L\Big\{x(t) \star y(t)\Big\} \Rightarrow Z(s) = X(s)\cdot Y(s) = \frac{1}{s+1}\cdot\frac{1}{s+2} \\ & \Rightarrow L^{-1}\Big\{Z(s)\Big\} = L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+2}\right\} \\ & \Rightarrow z(t) = L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-t} - e^{-2t} \end{aligned}$$

4. Consider the Laplace equation in polar form :
   $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, $0 < r < a$, $0 \le \theta < 2\pi$ subject to the condition $u(a,\theta) = f(\theta)$, where f is the given function. Let σ be the separation constant that appears when one uses the method of separation of variables. Then for solution $u(r,\theta)$ to be bounded and also periodic in $\theta$with period 2π, **NET(MS): (June)2013**
   (a) σ can not negative, (b) σ can be zero and in that case the solution is a constant
   (c)σ can be positive and in that case the solution must be an integer (d) the fundamental set of solutions is $\{1, r^n \sin n\theta, r^n \cos n\theta\}$, where n is a positive integer.
   Ans. (a), (b), (c) (d). (Note: All answers are correct.)

5. The differential equation
$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, u = u(x,t),\ 0 < x < \pi,\ t > 0$ with $u(0,t) = 0 = u(\pi,t),\ t > 0$
$u(x,0) = \sin x + \sin 2x,\ 0 \le x \le \pi$. Then
(a) $u(x,t) \to 0$ as $t \to 0$ for all $x \in (0,\pi)$
(b) $t^2 u(x,t) \to 0$ as $t \to 0$ for all $x \in (0,\pi)$
(c) $e^1 u(x,t)$ is a bounded function for $x \in (0,\pi),\ t > 0$
(d)$e^2(x,t) \to 0$ as $t \to 0$ for all $x \in (0,\pi)$ **NET(MS): (June)2012**
Ans. (a), (b), (c).

6. The solution of the ODE $\frac{d^2y}{dx^2} + y = 0,\ x > 0$ with $y(0) = 1,\ y'(0) = 0$ is equivalent to the Volterra integral equation **NET(MS): (Dec.)2012**
where (a) $y(x) = 1 + \int_0^x (t-x)y(t)dt$ (b) $y(x) = 1 + \int_0^x (t+x)y(t)dt$
(c) $y(x) = 1 + \int_0^x xty(t)dt$ (d) $y(x) = 1 + \int_0^x (x-t)y(t)dt$
Ans. (a).

7. Let $y(x)$ be a continuous solution of the initial value problem $y' + 2y = f(x),\ y(0) = 0$, where

$$\begin{aligned} f(x) &= 1,\ 0 \le x \le 1 \\ &= 0,\ x > 1 \end{aligned}$$

. Then $y(\frac{2}{3})$ is equal to **NET(MS): (June)2015**
(a) $\frac{\sinh(1)}{e^3}$ (b) $\frac{\cosh(1)}{e^3}$ (c) $\frac{\sinh(1)}{e^2}$ (d) $\frac{\cosh(1)}{e^2}$.
Ans. (c).

8. If Laplace transformation of $f(t)$ is $\frac{5}{s} + \frac{2s}{s^2+9}$, Then $f(0)$ is equals to
(A) 5 (B) 7 (C) 0 (D) ∞
Ans. (B)
Hint. By applying initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{5}{s} + \frac{2s}{s^2+9}\right) = 5 + 2 = 7$$

9. If Laplace transformation of $f(t)$ is $F(s) = \frac{2}{s(s+1)}$. Then $f(\infty)$ is equals to
A) 0 (B) 2 (C) 1 (D) $\propto$ **GATE(ECE)-03**
Ans. (B)
Hint. By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{2}{s(s+1)}\right) = 2$$

10. If $F(s) = \frac{2(s+1)}{s^2+4s+7}$. Then the initial and final values of $f(t)$ are respectively **GATE(ECE)-11**
(A) $0,2$ (B) $2,0$ (C) $0,2/7$ (D) $2/7,0$
Ans. (B)
Hint. By applying initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{2(s+1)}{s^2+4s+7}\right) = 2$$

By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{2(s+1)}{s^2+4s+7}\right) = 0$$

11. If $L[f(t)] = \frac{k}{s(s^2+4)}$. If $\lim_{t\to\infty} f(t) = 1$, then k is given by
(A) 4 (B) zero (C) $0 < k < 12$ (D) $5 < k < 12$
Ans. (A)
Hint. By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \Rightarrow \lim_{s\to 0} s\left(\frac{k}{s(s^2+4)}\right) = 1 \Rightarrow k = 4.$$

12. The Laplace transformation of $f(t)$ is given by $F(s) = \frac{2}{s(s+1)}$. As $t \longrightarrow \infty$, the value of $f(t)$ tends to **ECE-2003**
(A) 0 (B) 1 (C) 2 (D) ∞
Ans. (C)
Hint. By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{2}{s(s+1)}\right) = 2$$

13. Use Laplace transformation the value of $\int_0^\infty te^{-2t}\sin t dt$ is
A) $\frac{1}{25}$ B) $\frac{2}{25}$ C) $\frac{3}{25}$ D) $\frac{4}{25}$
Ans. (D)
Hint. Since $L\{\sin t\} = \frac{1}{s^2+1}$ and $L\{t\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \overline{f}(s)$. Now from the definition of Laplace transformation

$$\int_0^\infty e^{-st} f(t)dt = \overline{f}(s) \Rightarrow \int_0^\infty t\sin te^{-2t}dt = \overline{f}(2) = \frac{2\times 2}{(2^2+1)^2} = \frac{4}{25}$$

14. The inverse Laplace Transform of $\frac{s^2}{(s-3)^3}$ can be written as $\frac{e^{3t}}{2}[At^2+Bt+C]$. The values of A, B and C, respectively are **GATE(AE)-11**
(A) 3, 5 and 7 (B) 2, 10 and 12 (C) 10, 12 and 4 (D) 9, 12 and 2.
Ans. (D)
Hint. $L\{\frac{e^{3t}}{2}[At^2+Bt+C]\} = \frac{A}{(s-3)^3} + \frac{B}{2(s-3)^2} + \frac{C}{2(s-3)}$

15. The Green function G in x, t of the boundary value problem $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = 1$ with $y(0) = y(1) = 0$ is

$$\begin{aligned} G(x,t) &= f_1(x,t),\ x \le t \\ &= f_2(x,t),\ t \le x \end{aligned}$$

where **NET(MS): (Dec.)2011**
(a) $f_1(x,t) = -\frac{1}{2}t(1-x^2)$, $f_2(x,t) = -\frac{1}{2t}x^2(1-t^2)$
(b) $f_1(x,t) = -\frac{1}{2x}t^2(1-x^2)$, $f_2(x,t) = -\frac{1}{2t}x^2(1-t^2)$
(c) $f_1(x,t) = -\frac{1}{2t}x^2(1-t^2)$, $f_2(x,t) = -\frac{1}{2t}t(1-x^2)$
(d) $f_1(x,t) = -\frac{1}{2t}x^2(1-t^2)$, $f_2(x,t) = -\frac{1}{2t}t^2(1-x^2)$.
Ans. (a) and (c).

16. If $f(t) = L^{-1}\left[\frac{3s+1}{s^3+4s^2+(k-3)s}\right]$. If $\lim_{t\to\infty} f(t) = 1$. Then value of k is **ECE-2010**
(A) 1 (B) 2 (C) 3 (D) 4
Ans. (B)
Hint. By applying final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \quad \Rightarrow \lim_{s\to 0} s\left(\frac{3s+1}{s^3+4s^2+(k-3)s}\right) = 1$$
$$\text{or} \quad \frac{1}{k-3} = 1 \quad \Rightarrow k = 2$$

17. The boundary value problem $\frac{d^2y}{dx^2} = f(x)$, $x \in (0,1)$ with $y(0) = y(1) = 0$ is given by $y(x) = \int_0^1 G(x,\xi)f(\xi)d\xi$ **NET(MS): (Dec.)2012**

where
$$(a)\ G(x,\xi) = x(\xi-1),\ x \le \xi \qquad (b)\ G(x,\xi) = x^2(\xi-1),\ x \le \xi$$
$$= \xi(x-1),\ x > \xi \qquad = \xi^2(x-1),\ x > \xi$$

$$(c)\ G(x,\xi) = x(\xi^2-1),\ x \le \xi \qquad (d)\ G(x,\xi) = \sin x(\xi-1),\ x \le \xi$$
$$= \xi(x^2-1),\ x > \xi \qquad = \sin \xi(x-1),\ x > \xi$$
Ans. (a).

18. The solution of the initial value problem
$$y'' + 2y' + 10y = 6\delta(t),\ \ y(0) = y'(0) = 0$$
Where $\delta(t)$ denotes the Dirac-delta function , is
(a) $2e^t \sin 3t$, (b) $6e^t \sin 3t$ (c) $2e^{-t} \sin 3t$, (d) $6e^{-t} \sin 3t$. **GATE(MA)-12**
Ans. (c).

19. Let $y(t)$ be the continuous function on $[0, \infty)$ whose Laplace Transform exists. If $y(t)$ satisfies
$$\int_0^t (1-\cos(t-u))y(u)du = t^4,$$
then $y(1)$ is equal to
(A) 20 (B) 24 (C) 28 (D) 30 **GATE(MA)-15**
Ans. (C)
Hint. Using convolution Theorem, we get, $L\{1-\cos t\}\cdot L\{y(t)\} = L\{t^4\} \Rightarrow (\frac{1}{s} - \frac{s}{s^2+1})Y(s) = \frac{24}{s^5}$. Using inverse Laplace transform, we get, $y(t) = 24t + 4t^3$.

20. Let $y(t)$ be the continuous function on $[0, \infty)$ if $y(t) = t(1 - 4\int_0^t y(x)dx) + 4\int_0^t xy(x)dx$, then $\int_0^{\frac{\pi}{2}} y(t)dt$ is equal to **GATE(MA)-16**
Ans. $\frac{1}{2}$.
Hint. Using Laplace Transformation, we get, $Y(s) = \frac{1}{s^2} + 4\frac{d}{ds}\left(\frac{Y(s)}{s}\right) + 4\frac{L\{ty(t)\}}{s} = \frac{1}{s^2} + 4\frac{Y'(s)}{s} - 4\frac{Y(s)}{s^2} - 4\frac{Y'(s)}{s} \Rightarrow Y(s) = \frac{1}{2}\cdot\frac{2}{s^2+4}$. Using inverse Laplace transform, we get, $y(t) = \frac{\sin 2t}{2}$.

21. The solution of the integral equation $y(x) = x + \int_0^x \sin(x-t)y(t)dt$, is **GATE(MA)-13**

(A) $x^2 + \frac{x^3}{3}$ (B) $x - \frac{x^3}{3!}$ (C) $x + \frac{x^3}{3!}$ (D) $x^2 + \frac{x^3}{3!}$.
Ans. (C)

22. Consider the integral equation $y(x) = x^3 + \int_0^x \sin(x-t)y(t)dt$, $x \in [0, \pi]$. Then the value of $y(1)$ is **NET(JUNE)-16**
(A) $\frac{19}{20}$ (B) 1 (C) $\frac{17}{20}$ (D) $\frac{21}{20}$.
Ans. (D)

23. Let $y_1(x)$ and $y_2(x)$ be solutions of

$$x^2y'' + y' + (\sin x)y = 0$$

which satisfy the boundary conditions $y_1(0) = 0$, $y_1'(1) = 1$ and $y_2(0) = 1$, $y_2'(1) = 0$ respectively. Then, **GATE(MA)-03**
A) y_1 and y_2 do not have common zeros B) y_1 and y_2 have common zeros
C) either y_1 or y_2 has a zero of order 2 D) both y_1 and y_2 have zeros of order 2
Ans. B)

24. The initial value problem

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0,\ y(0) = 1,\ \left(\frac{dy}{dx}\right)_{x=0} = 0$$

has **GATE(MA)-06**

A) a unique solution B) no solution
C) infinitely many solution D) two linearly independent solutions.
Ans. B)

2.6 Review Exercises

1. Prove that $L^{-1}\left\{\frac{1}{(s-4)^2(s+3)}\right\} = 3te^{4t} - \frac{1}{7}(e^{4t} - e^{-37})$.
2. Prove that $L^{-1}\left\{\frac{4s+5}{(s^2+9)^2}\right\} = \frac{t\sin t}{2}$.
3. Using Convolution theorem, prove that $\int_0^t \sin x\cos(t-x)dx = \frac{t\sin t}{2}$
4. Prove that $L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{b\sin at - a\sin bt}{ab(b^2-a^2)}$
5. Find the boundary solution of $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$, $x > 0$, $t > 0$ given that $y(0, t) = 1$ and $y(x, 0) = 0$.
Ans. $y(x,t) = erfc\left(\frac{x}{2\sqrt{2}}\right)$
6. Find the boundary solution of $\frac{\partial y}{\partial x} - \frac{\partial y}{\partial t} = 1 - e^{-t}$, $0 < x < 1$, $t > 0$ given that $y(x, 0) = x$.
Ans. $y(x,t) = x + 1 - e^{-t}$
7. Show that the integral of the equation $(D^2 + 1)y = 0$ with $y(0) = 1$, $Dy(0) = 0$ is given by $y(t) = \cos\, t$ where $D \equiv \frac{d}{dt}$.

8. Solve $(D^2 - 4D + 4)x - y = 0,\ (D^2 + 4D + 4)y - 25x = 16e^t$.
Ans. $x = c_1e^{3t} + c_2e^{-3t} + c_3 \cos\, t + c_4 \sin\, t - e^t,\ y = c_1e^{3t} + 25c_2e^{-3t} + (3c_3 - 3c_4)\cos\, t + (3c_4 + 4c_3)\sin\, t - e^t$
9. Show that the integral of the equations $Dx + 2y = 0,\ Dy = x$ is given by $x^2 + y^2 + 2c = 0$. C.U(Hons.)-1989.
10. $(D^2 + 1)x + (D + 1)y = t,\ 2x + (D + 1)y = 0$, given that $x = y = 0$ at $t = 0$. B.U(Hons.)-1999
Ans. $x = -2e^t + 2e^{-t} - t,\ y = 2(e^t - 2te^{-t} + t - 1)$.
11. Use Laplace transform to solve $\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$, where $u(\frac{\pi}{2}, t) = 0, \frac{\partial u}{\partial x}_{x=0} = 0$ and $u(x, 0) = 30 \sin 5x$.
Ans. $u(x, t) = 30 \cos 5xe^{-70t}$.
12. Use Laplace transform to solve $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2},\ t > 0, x > 0$ where $u(x, 0) = 0, \frac{\partial u(x,0)}{\partial t} = 0$, $u(x, t) = 0$ as $x \to \infty$ and $u(0, t) = 0$.
Ans. $u(x, t) = (t - \frac{x}{a})H(t - \frac{x}{a})$.
13. Use Laplace transform to solve $\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = xt$ where $y(x, 0) = 0 = \frac{\partial y(x,0)}{\partial t}$ and $y(0, t) = 0$.
Ans. $y(x, t) = -\frac{xt^3}{6}$
14. Use Laplace transform to solve $\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}$ where $y(0, t) = 0,\ y(2, t) = 0$, and $y(x, 0) = 5 \sin\, 2\pi x,\ \frac{\partial y(x, 0)}{\partial t} = 0$.
Ans. $y(x, t) = 5 \sin 2\pi x \cos 6\pi t$
15. Use Laplace transform to solve $\frac{\partial y}{\partial x} = y + 2\frac{\partial y}{\partial t}$ given that $y(x,\ 0) = 6e^{-3x}$ which is bounded for $x > 0,\ t > 0$.
Ans. $y(x, t) = e^{-(3x+2t)}$
16. Use Laplace transform to solve $\frac{d^2 y}{dx^2} + 4y = \sin 2x$ given that $y(0) = 1,\ \frac{dy(0)}{dx} = 0$.
Ans. $y(x) = \cos 2x + \frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x$.
17. Use Laplace transform to solve $x^2\frac{d^2 y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$ given that $y(1) = 1,\ \frac{dy(1)}{dx} = 0$.
Ans. $y = (1 - 3\log x)x^2 + x^2(\log x)^2$.
18. Use Laplace transform to solve $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$ given that $y(0) = 1,\ \frac{dy(0)}{dx} = 0$.
Ans. $y = (1 - x)e^x + \frac{1}{6}x^3e^x$.
19. Use Laplace transform to solve $x^2\frac{d^2 y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ given that $y(1) = 0,\ \frac{dy(1)}{dx} = 0$.
Ans. $y(x) = -\frac{e}{x} + \frac{e^x}{x^2}$.
20. Use Laplace transform to solve $y'' - 4y' + 3y = 6t - 8$ with initial conditions $y(0) = y'(0) = 0$.
Ans. $y = 2t + e^t - e^{3t}$
21. Use Laplace transform to solve $(D^2 - 7D + 6)y = 2e^{3x}$ with $y = 1, \frac{dy}{dx} = 0$ at $x = 0$.
Ans. $y = \frac{7}{5}e^x - \frac{1}{15}e^{6x} - \frac{1}{3}e^{3x}$.

Chapter 3

Series Solution of Ordinary Differential Equations

3.1 Introduction

Various analytical methods have been discussed so far for solving ordinary differential equations to obtain exact solutions. However, in applied mathematics, science, and engineering applications, there are a large number of differential equations, especially those with variable coefficients, that cannot be solved exactly in terms of elementary functions, such as exponential, logarithmic, and trigonometric functions. For many of these differential equations, it is possible to find solutions in terms of infinite series. The main objective of this chapter is to present the essential techniques for solving such ordinary differential equations, in particular second-order linear ordinary differential equations with variable coefficients. A linear differential equation of order 2 can be written as

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = 0 \tag{3.1}$$

Where $p_0(x), p_1(x)$ are functions in x.

3.2 Review of Power Series

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots \tag{3.2}$$

where $a_0, a_1, a_2, a_3 \cdots$ are constants and x_0 is a fixed number. This series usually arises as the Taylor series of some function $f(x)$. If $x_0 = 0$, the power series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

3.3 Convergence of a Power Series

Power series (3.2) is convergent at x_0 if the

$$\lim_{N\to\infty}\sum_{n=0}^{N} a_n(x-x_0)^n$$

exists and finite. Otherwise, the power series is divergent. A power series will converge for some values of x and may diverge for other values. Series (3.2) is always convergent at $x = x_0$. If power series (3.2) is convergent for all x in the interval $|x - x_0| < r$ and is divergent whenever $|x - x_0| > r$ where $0 \le r < \infty$, then r is called the *radius of convergence* of the power series. Therefore, the radius of convergence r is given by

$$r = \frac{1}{\overline{\lim}|a_n|^{\frac{1}{n}}} \text{ or } r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

if this limit exists. Three very important power series are

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n\,, \quad -1 < x < 1,\\
e^x &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\,, \quad -\infty < x < \infty,\\
\cos x &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\,, \quad -\infty < x < \infty.
\end{aligned}$$

3.4 Operations of Power Series

Suppose functions $f(x)$ and $g(x)$ can be expanded into power series as

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n(x-x_0)^n\,, \quad \text{for } |x-x_0| < r_1,\\
g(x) &= \sum_{n=0}^{\infty} b_n(x-x_0)^n\,, \quad \text{for } |x-x_0| < r_2.
\end{aligned}$$

Then, for $|x - x_0| < r,\ r = \min(r_1, r_2)$,

$$f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-x_0)^n\,,$$

i.e., the power series of the sum or difference of the functions can be obtained by termwise addition and substraction. For multiplication,

$$f(x)g(x) = \Big[\sum_{n=0}^{\infty} a_n(x-x_0)^n\Big]\Big[\sum_{m=0}^{\infty} b_m(x-x_0)^n\Big] = \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} a_n b_{n-m}\Big)(x-x_0)^n$$

and for division

$$\frac{f(x)}{g(x)} = \frac{\sum_{n=0}^{\infty} a_n (x-x_0)^n}{\sum_{m=0}^{\infty} b_m (x-x_0)^n} = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

$$\text{where} \quad \sum_{n=0}^{\infty} a_n (x-x_0)^n = \Big[\sum_{n=0}^{\infty} b_n (x-x_0)^n\Big]\Big[\sum_{n=0}^{\infty} c_n (x-x_0)^n\Big],$$

in which c_n can be obtained by expanding the right-hand side and comparing coefficients of $(x-x_0)^n, n = 0, 1, 2, \cdots$.
If the power series of $f(x)$ is convergent in the interval $|x-x_0| < r_1$, then $f(x)$ is continuous and has continuous derivatives of all orders in this interval. The derivatives can be obtained by differentiating the power series termwise

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}, \quad \text{for } |x-x_0| < r_1,$$

The integral of $f(x)$ can be obtained by integrating the power series termwise

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n (x-x_0)^{n+1}}{n+1} + C, \quad \text{for } |x-x_0| < r_1,$$

3.5 Analytic Function

Definition 3.1 (**Analytical Function**) A function f defined in the interval I containing x_0 is said to be analytic at x_0 if $f(x)$ can be expressed as a power (Taylor) series $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, which has a positive radius of convergence.

Example 3.1 Prove that if a function f defined in the interval I containing x_0 is said to be analytic at x_0, then $\lim_{x \to x_0} f(x)$ exist and finite.

Proof: Since the function $f(x)$ is analytic at x_0, then $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \ |x-x_0| < r$, for some $r > 0$. So, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0$. Hence $\lim_{x \to x_0} f(x)$ exist and finite.

Example 3.2 Determine the radius of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{n}$.

Solution: $r = \lim_{n \to \infty} |\frac{a_n}{a_{n+1}}| = \lim_{n \to \infty} |\frac{n+1}{n}| = 1$.

Example 3.3 Determine the radius of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution: $r = \lim_{n \to \infty} |\frac{a_n}{a_{n+1}}| = \lim_{n \to \infty} |\frac{(n+1)!}{n!}| = \infty$.

3.6 Ordinary Point

Consider the n^{th}-order linear ordinary differential equation

$$y^n(x) + p_{n-1}(x)y^{n-1}(x) + p_{n-2}(x)y^{n-2}(x) + \cdots + p_0(x)y(x) = f(x).$$

Definition 3.2 (**Ordinary Point**) A point x_0 is called an ordinary point of the given differential equation if each of the coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ and $f(x)$ are analytic at $x = x_0$, i.e., $p_i(x)$, for $i = 0, 1, \cdots, n-1$, and $f(x)$ can be expressed as power series about x_0 that are convergent for $|x - x_0| < r,\ r > 0$, i.e.,

$$p_i(x) = \sum_{n=0}^{\infty} p_{i,n}(x - x_0)^n\ ,\quad f(x) = \sum_{n=0}^{\infty} f_n(x - x_0)^n.$$

3.7 Singularity at finite Point

Consider the n^{th}-order linear homogeneous ordinary differential equation

$$y^n(x) + p_{n-1}(x)y^{n-1}(x) + p_{n-2}(x)y^{n-2}(x) + \cdots + p_0(x)y(x) = 0.$$

Definition 3.3 (**Singular Point**)A point x_0 is called a singular point of the given differential equation if it is not an ordinary point, i.e., not all of the coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ are analytic at $x = x_0$.

Definition 3.4 (**Regular Singular Point**)A point x_0 is a regular singular point of the given differential equation if it is not an ordinary point (i.e., not all of the coefficients $p_k(x)$ are analytic) but all of $(x - x_0)^{n-k} p_k(x)$ are analytic for $k = 0, 1, \cdots, n-1$.

Definition 3.5 (**Irregular Singular Point**) A point x_0 is an irregular singular point of the given differential equation if it is neither an ordinary point nor a regular singular point.

Alternative Text for second order ODEs

A point $x = x_0$ of the differential equation (3.1) is called ordinary point then all $\lim_{x \to x_0} p_k(x),\ k = 0, 1$ are exist and finite. Otherwise the said point is called singular point.
A singular point $x = x_0$ of the differential equation (3.1) is called regular singular point if all $\lim_{x \to x_0} (x - x_0)^{2-k} p_k(x),\ k = 0, 1$ are exist and finite. A singular point which is not regular called irregular singular point.

Example 3.4 Discuss the ordinary and singular point of the differential equation

$$2x^2 \frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0.$$

Solution: The given differential equation $2x^2 \frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0$, can be written as $\frac{d^2y}{dx^2} + \frac{7x(x+1)}{2x^2}\frac{dy}{dx} - \frac{3}{2x^2}y = 0$. The given differential equation is compare with the differential equation (3.1) then $p_1(x) = \frac{7x(x+1)}{2x^2}$ and $p_0(x) = -\frac{3}{2x^2}$. Since neither $\lim_{x \to 0} p_1(x)$ nor $\lim_{x \to 0} p_0(x)$ does exist. So $p_1(x)$ and $p_0(x)$ are not analytic at $x = 0$. Hence, $x = 0$ is the singular point of the said differential equation. Now $\lim_{x \to x_0}(x - x_0)p_1(x) = \lim_{x \to 0}(x - 0)\frac{7x(x+1)}{2x^2} = \frac{7}{2}$ and $\lim_{x \to x_0}(x - x_0)^2 p_0(x) = \lim_{x \to 0}(x - 0)^2 \left(\frac{-3}{2x^2}\right) = -\frac{3}{2}$ are both exist and finite so the point $x = 0$ is a regular singular point. All points x $(x \neq 0)$ are ordinary points.

3.8 Singularity at infinity

Singularity Test at Infinity: To determine whether the point at infinite is a singular point or not, we transform the equation by $x = \frac{1}{t}$. Then

$$\frac{dy}{dx} = -t^2 \frac{dy}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt}$$

Then the differential equation (3.1) becomes

$$\frac{d^2y}{dt^2} + \left(\frac{2}{t} - \frac{1}{t^2} p_1(t^{-1})\right)\frac{dy}{dt} + \frac{1}{t^4} p_0(t^{-1}) = 0 \tag{3.3}$$

If $t = 0$ is a singular point of equation (3.3) then the origin equation (3.1) has a singularity at $x = \infty$.

Example 3.5 Show that the equation

$$\frac{d^2y}{dx^2} - \frac{2x}{1 - x^2}\frac{dy}{dx} + \frac{n(n+1)}{1 - x^2} y = 0 \tag{3.4}$$

has a singularity at $x = \infty$.

Solution: Substituting $x = \frac{1}{t}$ to the given equation. Then

$$\frac{dy}{dx} = -t^2 \frac{dy}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt}$$

Using these substitution the given equation reduces to

$$t^4 \frac{d^2y}{dt^2} + \frac{2t^3}{t^2 - 1}\frac{dy}{dt} + \frac{n(n+1)t^2}{t^2 - 1} y = 0 \tag{3.5}$$

Since $t = 0$ is a singular point of equation (3.5), so $x = \infty$ is a singular point of the given differential equation (3.4).

3.9 Series Solution about an Ordinary Point

Theorem 3.1 (Existence Theorem for Analytic Coefficients) Let x_0 be any real number and suppose that the coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ in

$$f(D)y = y^n(x) + p_{n-1}(x)y^{n-1}(x) + p_{n-2}(x)y^{n-2}(x) + \cdots + p_0(x)y(x)$$

have convergent power series expansions in powers of $x - x_0$ on an interval

$$|x - x_0| < r, \quad r > 0.$$

If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are any n constants, there exists a solution ϕ of the problem

$$f(D)y = 0, \; y(x_0) = \alpha_1, \; y'(x_0) = \alpha_2, \; \cdots, \; y^{(n-1)}(x_0) = \alpha_n,$$

with a power series expansion

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

convergent for $|x - x_0| < R$ where the radius of convergence $R \geq r$.

Theorem 3.2 Suppose that x_0 is an ordinary point of the n^{th}-order linear ordinary differential equation

$$y^n(x) + p_{n-1}(x)y^{n-1}(x) + p_{n-2}(x)y^{n-2}(x) + \cdots + p_0(x)y(x) = f(x).$$

where the coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ and $f(x)$ are analytic at $x = x_0$ and each can be expressed as a power series about x_0 convergent for $|x - x_0| < r, \; r > 0$. Then every solution of this differential equation can be expanded in one and only one way as a power series in $(x - x_0)$

$$y_i(x) = \sum_{n=0}^{\infty} a_{i,n} (x - x_0)^n, \; |x - x_0| < R$$

where the radius of convergence $R \geq r$.

In particular, the series solution about the ordinary point $x = x_0$ of second order ODE

Theorem 3.3 Suppose that x_0 is an ordinary point of the second order linear ordinary differential equation (3.1), i.e., the coefficients $p_0(x), p_1(x)$ are analytic at $x = x_0$ and then it has two non-trivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n, \; |x - x_0| < R, \tag{3.6}$$

for some $R > 0$, where a_n are constants and these power series converge in some interval $|x - x_0| < R$, $(R > 0)$ about x_0 (R being the radius of convergence of the power series).

Proof.: In order to get the coefficient a_n's in (3.6) we take

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \; |x - x_0| < R \tag{3.7}$$

Differentiating twice in a succession (3.7) gives

$$y'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}, \ |x-x_0| < R$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}, \ |x-x_0| < R.$$

Putting these values y, y' and y'' in (3.1) we get an equation of the form

$$A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots + A_n(x-x_0)^n + \cdots = 0 \tag{3.8}$$

where the coefficients $A_0, A_1, \cdots, A_n, \cdots$ are some function of the coefficient of $a_0, a_1, \cdots, a_n, \cdots$. Since (3.8) is an identity, all the coefficient $A_0, A_1, \cdots, A_n, \cdots$ of (3.8) must be zero, i.e

$$A_0 = 0, A_1 = 0, \cdots, A_n = 0, \cdots \tag{3.9}$$

Solving (3.9) we obtain the coefficient of (3.6) in terms of a_0 and a_1. Substituting the coefficients in (3.7) we get two independent series solution of (3.1) in powers of $(x-x_0)$ in $|x-x_0| < R$.

Theorem 3.4 The power series representation $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ about an ordinary point $x = x_0$ of the differential equation $a_0(x)\frac{d^2y(x)}{dx^2} + a_1(x)\frac{dy(x)}{dx} + a_2(x)y(x) = 0$ always converges. The maximum possible radius of convergence R is the distance from x_0 to the nearest singular point of the differential equation and the interval of convergence is $(R-x_0, R+x_0)$.

Example 3.6 Solve in series the equation

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \tag{3.10}$$

Solution: The given differential equation can be written as

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{y}{x^2+1} = 0 \tag{3.11}$$

Comparing the above differential equation with (3.1) we have $p_1(x) = \frac{x}{x^2+1}$ and $p_0(x) = -\frac{1}{x^2+1}$. Since, all the coefficients $p_0(x)$ and $p_1(x)$ are analytic at $x = 0$, i.e., $p_i(x)$ for $i = 0, 1$ can be expressed as power series about $x = 0$ that are convergent for $-1 < x < 1$, i.e. for $i = 0, 1$,

$$p_i(x) = (-1)^{i+1}x^i(1+x^2)^{-1} = (-1)^{i+1}x^i(1-x^2+x^4-x^6+\cdots), \quad -1 < x < 1.$$

So $x = 0$ is the ordinary point of the said differential equation and let

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} a_n x^n, \ -1 < x < 1 \tag{3.12}$$

be the series solution of (3.11). Differentiating twice in a succession (3.12) gives

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}, \ -1 < x < 1$$

$$\text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}, \ -1 < x < 1$$

Putting these values y, $y^{'}$ and $y^{''}$ in (3.10) we get

$$
\begin{aligned}
&(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \\
\Rightarrow\ &\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \\
\Rightarrow\ &\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0
\end{aligned}
$$

We shift the index of summation in the second series by 2, replacing n with $n+2$ and using the initial value $n=0$, and we shift the index of summation in the third series by 1, replacing n with $n+1$ and using the initial value $n=0$.

$$
2a_2 - a_0 + 6a_3 x + \sum_{n=2}^{\infty} \{n(n-1)a_n + (n+1)(n+2)a_{n+2} + na_n - a_n\} x^n = 0
$$

Equating the coefficients of various power of x to zero, we get

$$
\begin{aligned}
&2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}, \qquad 6a_3 = 0 \Rightarrow a_3 = 0, \\
&\text{and } n(n-1)a_n + (n+1)(n+2)a_{n+2} + na_n - a_n = 0 \\
\Rightarrow\ &a_{n+2} = -\frac{n-1}{n+2} a_n \text{ for all } n \geq 2
\end{aligned}
$$

Now putting $n = 2, 3, 4, \cdots$ successively in the above recurrence relation we get

$$
\begin{aligned}
&a_4 = -\frac{1}{4}a_2 = -\frac{1}{8}a_0, \qquad a_5 = -\frac{2}{3}a_3 = 0, \qquad a_6 = -\frac{1}{2}a_4 = \frac{1}{16}a_0 \\
&a_7 = -\frac{4}{7}a_5 = 0, \qquad a_8 = -\frac{5}{8}a_6 = -\frac{5}{128}a_0 \text{ and so on}
\end{aligned}
$$

Substituting the values of $a_0, a_1, a_2, \cdots$ in (3.12) we get the required solution as

$$
y(x) = a_0\{1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \cdots\} + a_1 x, \quad -1 < x < 1
$$

where a_0 and a_1 are arbitrary constants.

Example 3.7 Find a power series solution of the equation

$$
(x^2-1)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + xy = 0 \tag{3.13}
$$

Given that $y(0) = 4$ and $y^{'}(0) = 6$.

Solution: The given differential equation can be written as

$$
\frac{d^2y}{dx^2} + \frac{3x}{x^2-1}\frac{dy}{dx} + \frac{xy}{x^2-1} = 0 \tag{3.14}
$$

Comparing the above differential equation with (3.1) we have, $p_1(x) = \frac{3x}{x^2-1}$ and $p_0(x) = \frac{x}{x^2-1}$. Since, all the coefficients $p_0(x)$ and $p_1(x)$ are analytic at $x = 0$, i.e., $p_i(x)$ for $i = 0, 1$ can be expressed as power series about $x = 0$ that are convergent for $-1 < x < 1$, i.e. for $i = 0, 1$,

$$p_i(x) = -3^i x(1 - x^2)^{-1} = -3^i x(1 + x^2 + x^4 + x^6 + \cdots), \quad -1 < x < 1.$$

So $x = 0$ is the ordinary point of the said differential equation. Since the initial value of (3.13) are prescribed at $x = 0$ and $x = 0$ is an ordinary point, hence we shall find the required solution near $x = 0$, i.e in powers of x. So let

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n, \; -1 < x < 1 \tag{3.15}$$

be the series solution of (3.14). Differentiating twice in a succession (3.15) gives

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1}, \; -1 < x < 1 \qquad (3.16)\\ \text{and } y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \; -1 < x < 1 \end{aligned}$$

Putting these values y, y' and y'' in (3.13) we get

$$\begin{aligned} &(x^2 - 1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0 \\ \Rightarrow \; &\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \\ \Rightarrow \; &\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2} x^n + 3\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \end{aligned}$$

We shift the index of summation in the second series by 2, replacing n with $n + 2$ and using the initial value $n = 0$. We shift the index of summation in the fourth series by -1, replacing n by $n - 1$ and using the initial value $n = 1$.

$$-2a_2 - 6a_3 x + 3a_1 x + a_0 x + \sum_{n=2}^{\infty} \Big\{ n(n-1)a_n - (n+1)(n+2)a_{n+2} + 3n a_n + a_{n-1} \Big\} x^n = 0.$$

Equating the coefficients of various power of x to zero, we get

$$a_2 = 0, \qquad -6a_3 + 3a_1 + a_0 = 0 \Rightarrow a_3 = \frac{3a_1 + a_0}{6}$$

$$\text{and } n(n-1)a_n - (n+1)(n+2)a_{n+2} + 3n a_n + a_{n-1} = 0 \;\; \Rightarrow a_{n+2} = \frac{n(n+2)a_n + a_{n-1}}{(n+1)(n+2)} \;\; \forall n \geq 2$$

This last condition is called Recurrence formula. Given that $y(0) = 4$ and $y'(0) = 6$. Hence putting $x = 0$ in (3.15) and (3.16) we get

$$\begin{aligned} & a_0 = 4, \qquad a_1 = 6 \\ \text{and} \qquad & a_3 = \frac{3a_1 + a_0}{6} = \frac{3 \times 6 + 4}{6} = \frac{11}{3}. \end{aligned}$$

Now putting $n = 2, 3, 4, \cdots$ successively in the above recurrence formula we get

$$a_4 = \frac{8a_2 + a_1}{12} = \frac{8 \times 0 + 6}{12} = \frac{1}{2}, \qquad a_5 = \frac{15a_3 + a_2}{20} = \frac{15 \times \frac{11}{3} + 0}{20} = \frac{11}{4}$$
$$a_6 = \frac{24a_4 + a_3}{30} = \frac{24 \times \frac{1}{2} + \frac{11}{3}}{30} = \frac{47}{90} \text{ and so on}$$

Substituting the values of $a_0, a_1, a_2, \cdots$ in (3.15) the required solution is

$$y(x) = 4 + 6x + \tfrac{11}{3}x^3 + \tfrac{1}{2}x^4 + \tfrac{11}{4}x^5 + \tfrac{47}{90}x^6 + \cdots, \; -1 < x < 1.$$

Example 3.8 Solve $2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ in powers of $(x - 1)$.

Solution: The given differential equation is

$$2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \tag{3.17}$$

Since $x = 1$ is an ordinary point so let

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_n(x - 1)^n, \; -\infty < x < \infty \tag{3.18}$$

be the series solution of (3.17). We can simplify the calculation of the coefficient by translating the center of the expansion from $x = 1$ to $t = 0$. This is accomplished by the substitution $x - 1 = t$ or $x = t + 1$, then $\frac{dt}{dx} = 1$ or $dt = dx$. Now

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{dy}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{dy}{dt}) = \frac{d}{dt}(\frac{dy}{dt})\frac{dt}{dx} = \frac{d^2y}{dt^2}.$$

So the differential equation (3.17) transform to

$$2\frac{d^2y}{dt^2} + (t + 1)\frac{dy}{dt} + y = 0 \tag{3.19}$$

So (3.18) also transform to

$$y(t) = \sum_{n=0}^{\infty} a_n t^n, \; -\infty < t < \infty \tag{3.20}$$

Differentiating twice in a succession (3.20) gives

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Putting these values y, $y'(t)$ and $y''(t)$ in (3.20) we get

$$2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + (t + 1)\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$$
$$\Rightarrow \quad 2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^n + \sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$$
$$\Rightarrow \quad 2\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2} t^n + \sum_{n=1}^{\infty} n a_n t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1} t^n + \sum_{n=0}^{\infty} a_n t^n = 0$$

We shift the index of summation in the first series by 2, replacing n with $n+2$ and using the initial value $n=0$. We shift the index of summation in the third series by 1, replacing n by $n+1$ and using the initial value $n=0$.
Since we want to express everything in only one summation sign, we have to start the summation at $n=1$ in every series,

$$4a_2 + a_1 + a_0 + \sum_{n=1}^{\infty}\{2(n+1)(n+2)a_{n+2} + (n+1)a_n + (n+1)a_{n+1}\}t^n = 0$$

Equating the coefficients of various power of t to zero, we get

$$4a_2 + a_1 + a_0 = 0 \Rightarrow a_2 = -\frac{a_1 + a_0}{4}$$

$$\text{and } 2(n+1)(n+2)a_{n+2} + (n+1)a_n + (n+1)a_{n+1} = 0 \Rightarrow a_{n+2} = -\frac{a_n + a_{n+1}}{2(n+2)} \quad \forall n \geq 1$$

This last condition is called Recurrence formula. Now putting $n = 1, 2, 3, 4, \cdots$ successively in the above recurrence formula we get,

$$\begin{aligned} a_3 &= -\frac{a_1 + a_2}{6} = -\frac{a_1 - \frac{a_1+a_0}{4}}{6} = -\frac{3a_1 - a_0}{24} \\ a_4 &= -\frac{a_2 + a_3}{8} = -\frac{-\frac{a_1+a_0}{4} - \frac{3a_1-a_0}{24}}{8} = \frac{9a_1 + 5a_0}{192} \quad \text{and so on} \end{aligned}$$

Substituting the values of $a_0, a_1, a_2, \cdots$ in (3.20) the required solution is

$$\begin{aligned} y(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots, \ -\infty < t < \infty \\ \Rightarrow y(t) &= a_0 + a_1 t - \frac{a_1 + a_0}{4}t^2 - \frac{3a_1 - a_0}{24}t^3 + \frac{9a_1 + 5a_0}{192}t^4 - \cdots, \ -\infty < t < \infty \\ \Rightarrow y(t) &= a_0(1 - \frac{1}{4}t^2 + \frac{1}{24}t^3 + \frac{5}{192}t^4 - \cdots) + a_1(t - \frac{1}{4}t^2 - \frac{1}{8}t^3 + \frac{9}{192}t^4) - \cdots, \\ &\quad \text{in } -\infty < t < \infty. \text{ Now putting } t = x-1, \text{ we get} \\ \Rightarrow y(x) &= a_0\left\{1 - \frac{1}{4}(x-1)^2 + \frac{1}{24}(x-1)^3 + \frac{5}{192}(x-1)^4 - \cdots\right\} \\ &\quad + a_1\left\{(x-1) - \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^3 + \frac{9}{192}(x-1)^4 - \cdots\right\}, \ -\infty < x < \infty \end{aligned}$$

where a_0 and a_1 are arbitrary constants.

3.10 Series solution about regular Singular point $x = x_0$ (Frobenius Method)

Theorem 3.5 If the point x_0 is a regular singular point of the differential equation $a_0(x)\frac{d^2y(x)}{dx^2} + a_1(x)\frac{dy(x)}{dx} + a_2(x)y(x) = 0$, then it has at least one non-trivial solution of the form $y(x) = |x - x_0|^m \sum_{n=0}^{\infty} c_n(x-x_0)^n$, and this solution is valid in some interval $0 < |x - x_0| < R$ (where m is a certain (real or complex) constant and $R > 0$).

If $x = 0$, ($x_0 = 0$) is regular singular point, we shall use the Frobenius method to find the series solution about $x = 0$. Consider the differential equation of the form

$$\frac{d^2y}{dx^2} + \frac{P(x)}{x}\frac{dy}{dx} + \frac{Q(x)}{x^2}y = 0 \tag{3.21}$$

where the function $P(x)$ and $Q(x)$ are analytic for all $|x| < R$, $R > 0$. Then the following method of solving (3.21) is called Frobenius method. We assume a trial solution

$$\sum_{n=0}^{\infty} a_n x^{n+r}, a_0 \neq 0, \; 0 < x < R \tag{3.22}$$

In order to get the coefficient a_n's in (3.22) we take

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \; 0 < x < R \tag{3.23}$$

Differentiating twice in a succession (3.23) gives

$$y^{'}(x) = \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} \qquad \text{and} \qquad y^{''}(x) = \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-2}$$

Since $P(x)$ and $Q(x)$ are analytic at $x = 0$, we can write

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots \qquad \text{and} \qquad Q(x) = d_0 + d_1 x + d_2 x^2 + \cdots$$

Putting these values y, $y^{'}$, $y^{''}$, $P(x)$ and $Q(x)$ in (3.21) and then multiplying both sides by x^2, we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + (c_0 + c_1 x + \cdots)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + (d_0 + d_1 x + \cdots)\sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{3.24}$$

Since (3.24) is an identity, we can equate to zero the coefficient of various powers of x. This will give us a system of equations involving the unknowns coefficients a_n. The smallest power of x is x^r, and the corresponding equation is

$$[r(r-1) + c_0 r + d_0]a_0 = 0$$

Since by assumption $a_0 \neq 0$, we obtain

$$r^2 + (c_0 - 1)r + d_0 = 0$$

This important equation is known as **indicial equation** of (3.21). Solving this quadratic equation for r , one obtains two roots r_1 and r_2. Then there will be four different possibilities which are discussed in the following cases.

Case-I: Roots of the indicial equation unequal and not differ by an integer.
Let r_1 and r_2 be the roots of the indicial equation and $r_1 - r_2$ is not equal to an integer. Then the complete solution is given by
$y(x) = A(y(x))_{r=r_1} + B(y(x))_{r=r_2}$, $0 < x < R$ where A and B are arbitrary constants.

Case-II: Roots of the indicial equation equal.
Let r_1 and r_2 be the roots of the indicial equation and $r_1 = r_2$. Then the complete solution is given by
$y(x) = A(y(x))_{r=r_1} + B(\frac{\partial y(x)}{\partial r})_{r=r_1},\ 0 < x < R$ where A and B are arbitrary constant.

Case-III: Roots of the indicial equation unequal, differing by an integer and making a coefficient of y infinite.
Let r_1 and r_2 be the roots of the indicial equation are distinct and differ by an integer and if some of the coefficient of $y(x)$ becomes infinite when $r = r_1$, we modify the form of $y(x)$ by replacing a_0 by $b_0(r - r_1)$. We then obtain two independent solutions by putting $r = r_1$ in the modified form of $y(x)$ and $\frac{\partial y(x)}{\partial r},\ 0 < x < R$. The result of putting $r = r_2$ in $y(x)$ gives a numerical multiple of that obtained by putting $r = r_1$ and hence we reject the solution obtained by putting $r = r_2$ in $y(x)$.

Case-IV: Roots of the indicial equation unequal, differing by an integer and making a coefficient of y indeterminate.
Let r_1 and r_2 be the roots of the indicial equation are distinct and differ by an integer and if one of the coefficient of y becomes indeterminate when $r = r_2$. Then the complete solution is given by putting $r = r_2$ in $y(x),\ 0 < x < R$, which contain two arbitrary constants. The result of putting $r = r_1$ in $y(x)$ gives a numerical multiple of that obtained by putting $r = r_2$ and hence we reject the solution obtained by putting $r = r_1$ in y.

Note: If a series solution about a point $x = x_0 \neq 0$ is to be determined, one can change the independent variable x to $t = x - x_0$ and then solve the resulting differential equation about $t = 0$.

Illustrative Example:
Case-I: Roots of indicial equation unequal and not differ by an integer.

Example 3.9 Find the power series solution of the equation
$$2x^2y''(x) + xy'(x) - (x+1)y(x) = 0$$
in powers of x.

Solution: The given differential equation can be written as

$$y''(x) + \frac{1}{2x}y'(x) - \frac{x+1}{2x^2}y(x) = 0 \tag{3.25}$$

Comparing the above differential equation with (3.1) we have $p_1(x) = \frac{1}{2x}$ and $p_0(x) = \frac{x+1}{2x^2}$. Since the point $x = 0$ is the singular point. Now $\lim_{x\to x_0}(x - x_0)p_1(x) = \lim_{x\to 0}(x-0)\frac{x}{2x^2} = \frac{1}{2}$ and $\lim_{x\to x_0}(x - x_0)^2p_0(x) = \lim_{x\to 0}(x-0)^2\frac{-(x+1)}{2x^2} = -\frac{1}{2}$ are both exist and finite so the point $x = 0$ is a regular singular point i.e. both $xp_1(x)$ and $x^2p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < \infty$.
Let us assume that the trial solution of equation (3.25) is

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r}, a_0 \neq 0,\ 0 < x < \infty. \tag{3.26}$$

Differentiating twice in a succession (3.26) gives

$$y^{'}(x)=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} \qquad \text{and} \qquad y^{''}(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2},\ 0<x<\infty.$$

Putting these values y, $y^{'}$ and $y^{''}$ in (3.25) we get

$$\begin{aligned}
&2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+x\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-(x+1)\sum_{n=0}^{\infty}a_nx^{n+r}=0\\
\Rightarrow\ &2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1}-\sum_{n=0}^{\infty}a_nx^{n+r}=0\\
\Rightarrow\ &\sum_{n=0}^{\infty}\Big\{2(n+r)(n+r-1)+(n+r)-1\Big\}a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1}=0\\
\Rightarrow\ &\sum_{n=0}^{\infty}\Big\{(2n+2r+1)(n+r-1)\Big\}a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1}=0
\end{aligned}$$

Equating to zero the coefficient of smallest power of x, namely x^r, the indicial equation is $a_0(2r+1)(r+1)=0$ so that roots of the equation are $r=1$ and $-\frac{1}{2}$ as $a_0\neq 0$. Here the roots of the indicial equation are distinct and the difference is $=1-(-\frac{1}{2})=\frac{3}{2}$ which is not an integer. Now equating the coefficient of x^{n+r}, we obtain the recurrence relation as

$$(2n+2r+1)(n+r-1)a_n-a_{n-1}=0\Rightarrow a_n=\frac{a_{n-1}}{(2n+2r+1)(n+r-1)}.$$

Putting $n=1,2,3,\cdots$, we get $a_1=\frac{1}{r(2r+3)}a_0$, $a_2=\frac{1}{(2r+5)(r+1)}a_1=\frac{1}{(2r+5)(2r+3)r(r+1)}a_0$ and so on. Putting these values in (3.26), we get

$$y=a_0x^r\Big[1+\frac{x}{(2r+3)r}+\frac{x^2}{(2r+5)(2r+3)r(r+1)}+\cdots\Big] \tag{3.27}$$

Putting $r=1$ in (3.27), we get $(y(x))_{r=1}=a_0x\Big[1+\frac{1}{5}x+\frac{1}{70}x^2+\cdots,\ 0<x<\infty\Big]$. Next putting $r=-\frac{1}{2}$ in (3.26), we get $(y(x))_{r=-\frac{1}{2}}=a_0x^{-\frac{1}{2}}[1-x+\frac{1}{2}x^2+\cdots,\ 0<x<\infty]$. Hence the required solution is given by

$$y(x)=A(y(x))_{r=1}+B(y(x))_{r=-\frac{1}{2}},\ 0<x<\infty.$$

where A and B are two arbitrary constants.

Case-II: Roots of indicial equation are equal.

Example 3.10 Use Method of Frobenius to solve the following differential equation:

$$xy^{''}+y^{'}+xy=0$$

Solution: The given differential equation can be written as

$$y^{''}+\frac{y^{'}}{x}+y=0 \tag{3.28}$$

Comparing the above differential equation with (3.1) we have $p_1(x)=\frac{1}{x}$, $p_0(x)=1$. Obviously, $x=0$ is a singular point. Note that $xp_1(x)=1$ and $x^2p_0(x)=x^2$. Both $xp_1(x)$ and $x^2p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < \infty$. Hence, $x = 0$ is a regular singular point. Let us assume that the trial solution of equation (3.28) is

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r}, a_0 \neq 0,\ 0 < x < \infty \tag{3.29}$$

Differentiating twice in a succession (3.29) gives

$$y' = \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} \qquad \text{and} \qquad y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-2},\ 0 < x < \infty.$$

Putting these values y, y' and y'' in (3.28) we get

$$x\sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Equating to zero the coefficient of smallest power of x, namely x^{r-1}, the indicial equation is $a_0 r(r-1) + ra_0 = 0$ so that root of the equation is $r = 0$ and as $a_0 \neq 0$. Here the roots of the indicial equation are equal. Next equating to zero the coefficient of x^{n+r+1}, we obtain the recurrence relation as

$$(n + r + 2)(n + r + 1)a_{n+2} + (n + r + 2)a_{n+2} + a_n = 0 \Rightarrow a_{n+2} = -\frac{a_n}{(n + r + 2)^2},\ n = 0, 1, 2, \cdots \tag{3.30}$$

Next equating to zero, the coefficient of x^r, we get

$$a_1(r + 1)^2 = 0 \qquad \text{so that } a_1 = 0 \text{ (Since } r = 0 \text{ is a root of indicial equation).}$$

Using $a_1 = 0$ and (3.30), we get $a_1 = a_3 = a_5 = a_7 = \cdots = 0$. Putting $n = 0, 2, 4, \cdots$ in (3.30), we get

$$a_2 = -\frac{a_0}{(r + 2)^2}, \qquad a_4 = -\frac{a_2}{(r + 4)^2} = \frac{a_0}{(r + 2)^2(r + 4)^2}$$

$$a_6 = -\frac{a_4}{(r + 6)^2} = -\frac{a_0}{(r + 2)^2(r + 4)^2(r + 6)^2} \text{ and so on}$$

Putting these values in (3.29), we get

$$y(x) = a_0 x^r\left[1 - \frac{x^2}{(r + 2)^2} + \frac{x^4}{(r + 2)^2(r + 4)^2} - \frac{x^6}{(r + 2)^2(r + 4)^2(r + 6)^2} + \cdots\right] \tag{3.31}$$

Differentiating partially equation (3.31) with respect to r, we have

$$\begin{aligned}\frac{\partial y}{\partial r} &= a_0 x^r \log x\left\{1 - \frac{x^2}{(r + 2)^2} + \frac{x^4}{(r + 2)^2(r + 4)^2} - \frac{x^6}{(r + 2)^2(r + 4)^2(r + 6)^2} + \cdots\right\} \\ &+a_0 x^r\left[-\frac{x^2}{(r + 2)^2} \times \frac{(-2)}{r + 2} + \frac{x^4}{(r + 2)^2(r + 4)^2}\left\{\frac{-2}{r + 2} - \frac{2}{r + 4}\right\}\right. \\ &\left.-\frac{x^6}{(r + 2)^2(r + 4)^2(r + 6)^2}\left\{\frac{-2}{r + 2} - \frac{2}{r + 4} - \frac{2}{r + 6}\right\} + \cdots\right]\end{aligned} \tag{3.32}$$

Putting $r = 0$ and replacing a_0 by A in (3.31), we get

$$\left(y\right)_{r=0} = A[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots] = Au(x)(say)$$

Next putting $r = 0$ in (3.32) and replacing a_0 by B, we get

$$\begin{aligned}\left(\frac{\partial y}{\partial r}\right)_{r=0} &= Blogx\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots\right] \\ &+ B\left[\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}(1 + \frac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots\right] \\ &= B[u \log x + \{\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}(1 + \frac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}(1 + \frac{1}{2} + \frac{1}{3}) + \cdots\}] = Bv(x)(say)\end{aligned}$$

Hence the required solution is given by $y = Au(x) + Bv(x)$, $0 < x < \infty$ where A and B are arbitrary constants.

Case-III: Roots of indicial equation unequal differ by an integer and making a coefficient of y infinite.

Example 3.11 Find the power series solution of the equation

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

in powers of x.

Solution: The given differential equation can be written as

$$y'' + \frac{y'}{x} + \frac{(x^2 - 1)y}{x^2} = 0 \tag{3.33}$$

Comparing the above differential equation with (3.1) we have $p_1(x) = \frac{1}{x}, p_0(x) = \frac{x^2-1}{x^2}$. Obviously, $x = 0$ is a singular point. Note that $\lim_{x\to 0} xp_1(x) = 1$ and $\lim_{x\to 0} x^2 p_0(x) = -1$. Both $xp_1(x)$ and $x^2 p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < \infty$. Hence, $x = 0$ is a regular singular point. Let us assume that the trial solution of of equation (3.33) is

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r}, a_0 \neq 0,\ 0 < x < \infty \tag{3.34}$$

Differentiating twice in a succession (3.34) gives

$$y' = \sum_{n=0}^{\infty} (n + r)a_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r-2},\ 0 < x < \infty$$

Putting these values y, y' and y'' in (3.33) we get

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+x\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+(x^2-1)\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}-\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\Big[(n+r)(n+r-1)+(n+r)-1\Big]a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}=0$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\Big[(n+r-1)(n+r+1)\Big]a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}=0 \qquad (3.35)$$

Equating to zero the coefficient of smallest power of x, namely x^r in (3.35), we obtain the indicial equation is $a_0(r+1)(r-1)=0$ so that roots of the equation are $r=-1$ and 1 as $a_0\neq 0$. Here the roots of the indicial equation are unequal and differ by an integer. Next equating zero the coefficient of x^{n+r} in (3.35), we obtain the recurrence relation as

$$(n+r+1)(n+r-1)a_n+a_{n-2}=0\Rightarrow a_n=-\frac{a_{n-2}}{(n+r+1)(n+r-1)} \qquad (3.36)$$

Next equating the coefficient x^{r+1} in (3.35) and we get

$$a_1r(r+2)=0\Rightarrow a_1=0 \ \text{(Since both the roots of indicial equation are } r=1 \text{ and } r=-1).$$

Using $a_1=0$ and (3.36), we get $a_1=a_3=a_5=a_7=\cdots=0$. Putting $n=2,4,6,\cdots$, in (3.36) we get

$$a_2=-\frac{a_0}{(r+1)(r+3)},\ a_4=-\frac{a_2}{(r+3)(r+5)}=\frac{a_0}{(r+1)(r+3)^2(r+5)} \quad \text{and so on}$$

Putting these values in (3.34), we get

$$y=a_0x^r\Big[1-\frac{x^2}{(r+1)(r+3)}+\frac{x^4}{(r+1)(r+3)^2(r+5)^2}-\cdots\Big] \qquad (3.37)$$

Now if we take $r=-1$ in the above series, the coefficients become infinite because of the factor $(r+1)$ in the denominator. To get out of this difficulty we put $a_0=b_0(r+1)$ in (3.37) and get modified solution as

$$y=b_0x^r\Big[(r+1)-\frac{x^2}{(r+3)}+\frac{x^4}{(r+3)^2(r+5)}-\cdots\Big] \qquad (3.38)$$

Differentiating partially equation (3.38) with respect to r, we have

$$\frac{\partial y}{\partial r}=b_0x^r logx\Big\{(r+1)-\frac{x^2}{(r+3)}+\frac{x^4}{(r+3)^2(r+5)}-\cdots\Big\}$$

$$+b_0x^r\Big[1+\frac{x^2}{(r+3)^2}-\frac{x^4}{(r+3)^2(r+5)}\{\frac{2}{(r+3)^3(r+5)}+\frac{1}{(r+3)(r+5)^2}\}-\cdots\Big] \qquad (3.39)$$

Putting $r=-1$ and replacing b_0 by A in (3.39), we get

$$\Big(y\Big)_{r=-1}=Ax^{-1}\Big[-\frac{x^2}{2}+\frac{x^4}{2^2.4}-\cdots\Big]=Au(say) \qquad (3.40)$$

Next putting $r = -1$ in (3.40) and replacing b_0 by B, we get

$$\left(\frac{\partial y}{\partial r}\right)_{r=-1} = Bx^{-1}logx\Big[-\frac{x^2}{2}+\frac{x^4}{2^2.4}+\cdots\Big]+Bx^{-1}\Big[1+\frac{x^2}{2^2}-\frac{x^4}{2^2.4}\Big(\frac{1}{16}+\frac{1}{32}\Big)+\cdots\Big]$$

$$\Rightarrow \left(\frac{\partial y}{\partial r}\right)_{r=-1} = B\Big[ulogx+Bx^{-1}\Big\{1+\frac{x^2}{2^2}-\frac{x^4}{2^2.4}\Big(\frac{1}{16}+\frac{1}{32}\Big)+\cdots\Big\}\Big] = Bv(say)$$

Hence the required solution is given by $y = Au(x) + Bv(x)$, $0 < x < \infty$, where A and B are arbitrary constants.

Note: When $x = x_0$ is an ordinary point of the differential equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

then we can also solve this type of differential equation by Frobenius Method.

Case-IV: Roots of indicial equation unequal differ by an integer and making a coefficient of y indeteminate

Example 3.12 Find the power series solution of the equation

$$(1-x^2)y'' - xy' + 4y = 0$$

in powers of x.

Solution: The given differential equation can be written as

$$y'' - \frac{x}{1-x^2}y' + \frac{4}{1-x^2}y = 0 \tag{3.41}$$

Comparing the above differential equation with (3.1) we have $p_0(x) = \frac{4}{1-x^2}$, $p_1(x) = -\frac{x}{1-x^2}$ and $p_0(0) = 4$, $p_1(0) = 0$ so the point $x = 0$ is the ordinary point. Let us assume that the trial solution of equation (3.41) be

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r}, a_0 \neq 0,\ |x| < 1 \tag{3.42}$$

Differentiating twice in a succession (3.42) gives

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

Putting these values y, y' and y'' in (3.41) we get

$$(1-x^2)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 4\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$- \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + 4\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - \sum_{n=0}^{\infty}\Big[(n+r)(n+r-1)+(n+r)-4\Big]a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - \sum_{n=0}^{\infty}\Big[(n+r+2)(n+r-2)\Big]a_n x^{n+r} = 0$$

Equating to zero the coefficient of smallest power of x, namely x^{r-2},we obtain the indicial equation is $a_0 r(r-1) = 0$ so that roots of the equation are $r = 0$ and 1 as $a_0 \neq 0$. Here the roots of the indicial equation are unequal and differ by an integer. Next equating zero the coefficient of x^{n+r-2}, we obtain the recurrence relation as

$$(n+r)(n+r-1)a_n - (r+n)(r+n-4)a_{n-2} = 0 \Rightarrow a_n = \frac{(r+n-4)}{(n+r-1)}a_{n-2} \tag{3.43}$$

Next equating the coefficient x^{r-1} and get $a_1 r(r+1) = 0$. If we take $r = 0$, then a_1 is indeterminate with $r = 0$ and using (3.43), we can express $a_2, a_4, a_6, \cdots$ in terms of a_0 and $a_3, a_5, a_7, \cdots$ in terms of a_1 if we assume a_1 is finite. Thus at $r = 0$ (3.43) reduces to

$$a_n = \frac{(n-4)}{(n-1)}a_{n-2} \tag{3.44}$$

Putting $n = 2, 3, 4, 5, 6, \cdots$ in (3.44), we get

$$a_2 = -2a_0,\ a_3 = -\frac{1}{2}a_1 = -\frac{a_1}{2},\ a_4 = a_6 = a_8 = \cdots = 0,$$
$$a_5 = \frac{a_3}{4} = -\frac{a_1}{8},\ a_7 = \frac{3}{6}a_5 = -\frac{a_1}{16} \text{ and so on}$$

Putting $r = 0$ and $a_2, a_3, a_4, \cdots$ in (3.42), we get

$$y(x) = \left[a_0 + a_1 x - 2a_0 x^2 - \frac{a_1}{2}x^3 - \frac{a_1}{8}x^5 - \frac{a_1}{16}x^7 + \cdots,\ |x| < 1\right] \tag{3.45}$$

$$\Rightarrow y(x) = a_0\left(1 - 2x^2\right) + a_1\left(x - \frac{x^3}{2} - \frac{x^5}{8} - \frac{x^7}{16} - \cdots,\ |x| < 1\right) \tag{3.46}$$

which is the required solution, where a_0 and a_1 are two arbitrary constants.

3.11 Worked out Examples

Example 3.13 Find the series solution of ODE

$$y'' + xy' + x^2 y = 0$$

about the point $x = 0$.

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0 \tag{3.47}$$

Since $x = 0$ is an ordinary point so let

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n,\ 0 < x < \infty \tag{3.48}$$

be the trial solution of (3.47). Differentiating twice in a succession (3.48) gives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},\ 0 < x < \infty$$

Putting these values y, $y^{'}$ and $y^{''}$ in (3.47) we get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + x^2\sum_{n=0}^{\infty} a_n x^n = 0$$
$$\Rightarrow \quad \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

We shift the index of summation in the first series by 2, replacing n with $n+2$ and using the initial value $n = 0$. We shift the index of summation in the third series by 2, replacing n with $n-2$ and using the initial value $n = 2$. Since we want to express everything in only one summation sign, we have to start the summation at $n = 2$ in every series,

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$
$$\Rightarrow \quad 2a_2 + (6a_3 + a_1)x + \sum_{n=2}^{\infty}\{(n+1)(n+2)a_{n+2} + na_n + a_{n-2}\}x^n = 0$$

Equating the coefficients of various power of x to zero, we get

$$2a_2 = 0 \Rightarrow a_2 = 0, \qquad 6a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{6}$$
$$\text{and } (n+1)(n+2)a_{n+2} + na_n + a_{n-2} = 0 \Rightarrow a_{n+2} = -\frac{na_n + a_{n-2}}{(n+1)(n+2)} \quad \forall n \geq 2$$

This last condition is called recurrence formula. Now putting $n = 2, 3, 4, \cdots$ successively in the above recurrence formula we get

$$a_4 = -\frac{2a_2 + a_0}{12} = -\frac{a_0}{12}, \; a_5 = -\frac{3a_3 + a_1}{20} = -\frac{3}{20}a_3 - \frac{1}{20}a_1 = \frac{1}{40}a_1 - \frac{1}{20}a_1 = \frac{a_1}{40} \text{ and so on}$$

Substituting the values of $a_2, a_3, a_4, \cdots$ in (3.48) the required solution is

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ \Rightarrow y &= a_0 + a_1x - \frac{a_1}{6}x^3 - \frac{a_0}{12}x^4 + \frac{a_1}{40}x^5 + \cdots \\ \Rightarrow y &= a_0(1 - \frac{1}{12}x^4 + \cdots) + a_1(x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - \cdots), \; 0 < x < \infty \end{aligned}$$

Example 3.14 Find the series solution of the equation
$$\frac{d^2y}{dx^2} + y = 0$$
near $x = 0$ such that $y(0) = 1$, $y^{'}(0) = 2$.

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} + y = 0 \tag{3.49}$$

Comparing the above differential equation with (3.1) we have $p_1(x) = 0$ and $p_0(x) = 1$. Obviously, $x = 0$ is a ordinary point. Note that both $p_1(x)$ and $p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < \infty$. So let

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n, \; 0 < x < \infty \tag{3.50}$$

be the series solution of (3.49). Differentiating twice in a succession (3.50) gives

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \tag{3.51}$$

Putting these values y, y' and y'' in (3.49) we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

We shift the index of summation in the first series by 2, replacing n with $n + 2$ and using the initial value $n = 0$.

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\Big[(n+1)(n+2)a_{n+2} + a_n\Big]x^n = 0$$

Equating the coefficients of x^n to zero, we get

$$(n+1)(n+2)a_{n+2} + a_n = 0 \Rightarrow a_{n+2} = -\frac{1}{(n+1)(n+2)}a_n \;\text{ for all }\; n \geq 0.$$

Now putting $n = 0, 1, 2, 3, 4, \cdots$ successively in the above recurrence relation we get

$$a_2 = -\frac{1}{1.2}a_0 = -\frac{1}{2!}a_0,\; a_3 = -\frac{1}{3.2}a_1 = -\frac{1}{3!}a_1,\; a_4 = -\frac{1}{4.3}a_2 = \frac{1}{4.3.2!}a_0 = \frac{1}{4!}a_0$$
$$a_5 = -\frac{1}{5.4}a_3 = -\frac{1}{5.4.3!}a_1 = \frac{1}{5!}a_1,\; a_6 = -\frac{1}{6.5}a_4 = -\frac{1}{6.5.4!}a_0 = \frac{1}{6!}a_0 \;\text{ and so on}$$

Substituting the values of $a_2, a_3, a_4, \cdots$ in (3.50) we get

$$y = a_0\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\} + a_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) \Rightarrow y = a_0 \sin x + a_1 \cos x$$

Using the conditions $y(0) = 1$ and $y'(0) = 2$ in (3.52) and (3.51), we get $a_0 = 1$ and $a_1 = 2$. Hence the required solution is

$$y = \cos x + 2 \sin x, \; 0 < x < \infty$$

Example 3.15 Obtain the power series solution of the differential equation

$$2x^2 y'' + (2x^2 - x)y' + y = 0$$

near the point $x = 0$.

Solution: The given differential equation can be written as

$$y''(x) + \frac{(2x^2 - x)}{2x^2}y'(x) + \frac{y(x)}{2x^2} = 0 \tag{3.52}$$

Comparing the above differential equation with (3.1) we have $p_1(x) = \frac{2x^2-x}{2x^2}$ and $p_0(x) = \frac{1}{2x^2}$. Obviously, $x = 0$ is a singular point. Note that $xp_1(x) = \frac{2x-1}{2}$ and $x^2p_0(x) = \frac{1}{2}$. Both $xp_1(x)$ and $x^2p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < \infty$. Hence, $x = 0$ is a regular singular point. Let us assume that the trial solution of of equation (3.52) is

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r}, a_0 \neq 0, \ 0 < x < \infty \tag{3.53}$$

Differentiating twice in a succession (3.53) gives

$$y^{'} = \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} \qquad \text{and} \qquad y^{''} = \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-2}$$

Putting these values y, $y^{'}$ and $y^{''}$ in (3.52) we get

$$2x^2 \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-2} + (2x^2 - x)\sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \quad 2\sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r} + 2\sum_{n=0}^{\infty}(n + r)a_n x^{n+r+1} - \sum_{n=0}^{\infty}(n + r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}\Big[2(n + r)(n + r - 1) - (n + r) + 1\Big]a_n x^{n+r} + 2\sum_{n=0}^{\infty}(n + r)a_n x^{n+r+1} = 0$$

Equating to zero the coefficient of smallest power of x, namely x^r, we obtain the indicial equation as

$$a_0[2r(r - 1) - r + 1] = 0 \Rightarrow r = \frac{1}{2}, 1 \text{ as } a_0 \neq 0$$

Here the roots of the indicial equation are unequal and not differ by an integer. Next equating zero the coefficient of x^{n+r}, we obtain the recurrence relation as

$$(2n + 2r - 1)(n + r - 1)a_n + 2(n + r - 1)a_{n-1} = 0$$

$$\Rightarrow \quad a_n = -\frac{2a_{n-1}}{(2n + 2r - 1)} \tag{3.54}$$

Putting $n = 1, 2, 4, 6, \cdots$, in above recurrence relation we get

$$a_1 = -\frac{2a_0}{2r + 1}, \ a_2 = -\frac{2a_1}{2r + 3} = \frac{2^2 a_0}{(2r + 1)(2r + 3)}, \ a_3 = -\frac{2a_2}{2r + 5} = -\frac{2^3 a_0}{(2r + 1)(2r + 3)(2r + 5)}$$

and so on. Putting these values in (3.53), we get

$$y = a_0 x^r\Big[1 - \frac{2x}{2r + 1} + \frac{2^2x^2}{(2r + 1)(2r + 3)} - \frac{2^3x^3}{(2r + 1)(2r + 3)(2r + 5)} + \cdots\Big] \tag{3.55}$$

Putting $r = \frac{1}{2}$ and replacing a_0 by A in (3.55), we get

$$\Big(y\Big)_{r=\frac{1}{2}} = Ax^{\frac{1}{2}}\Big[1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\Big] = Au(say)$$

Next putting $r = 1$ in (3.55) and replacing a_0 by B, we get

$$\left(y\right)_{r=1} = Bx\left[1 - \frac{2}{3}x + \frac{4x^2}{15} - \frac{8x^3}{105} + \cdots\right] = Bv(say)$$

Hence the required solution is given by $y(x) = Au(x) + Bv(x),\ 0 < x < \infty$, where A and B are arbitrary constants.

Example 3.16 Use method of Frobenious to find solution of the differential equation

$$(x^2 - x)y'' + (3x - 1)y' + y = 0$$

Solution: The given differential equation can be written as

$$y''(x) + \frac{3x - 1}{x^2 - x}y'(x) + \frac{y(x)}{x^2 - x} = 0 \tag{3.56}$$

Comparing the above differential equation with (3.1) we have $p_1(x) = \frac{3x-1}{x^2-x}$ and $p_0(x) = \frac{1}{x^2-x}$. Obviously, $x = 0$ is a singular point. Note that $xp_1(x) = \frac{3x-1}{x-1}$ and $x^2p_0(x) = \frac{x}{x-1}$. Both $xp_1(x)$ and $x^2p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < 1$. Hence, $x = 0$ is a regular singular point. Let us assume that the trial solution of of equation (3.52) is

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r}, a_0 \neq 0,\ 0 < x < 1 \tag{3.57}$$

Differentiating twice in a succession (3.57) gives

$$y' = \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} \qquad \text{and} \qquad y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-2}$$

Putting these values y, y' and y'' in (3.56) we get

$$(x^2 - x)\sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-2} + (3x - 1)\sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-1} + 3\sum_{n=0}^{\infty}(n + r)a_n x^{n+r}$$

$$- \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\Big[(n + r)(n + r - 1) + 3(n + r) + 1\Big]a_n x^{n+r} + \sum_{n=0}^{\infty}\Big[(n + r) - (n + r)(n + r - 1)\Big]a_n x^{n+r-1} = 0$$

Equating to zero the coefficient of smallest power of x, namely x^{r-1},we obtain the indicial equation as $a_0r^2 = 0$ so that $r = 0$ as $a_0 \neq 0$. Here the roots of the indicial equation are equal. Next equating zero the coefficient of x^{n+r-1}, we obtain the recurrence relation as

$$(n + r + 1)^2 a_{n-1} - (n + r)^2 a_n = 0, \qquad \Rightarrow a_n = \frac{(n + r + 1)^2}{(n + r)^2}a_{n-1}$$

Putting $n = 1, 2, 3, 4, \cdots$, in above recurrence relation, we get

$$a_1 = -\frac{(r+2)^2}{(r+1)^2}a_0,\ a_2 = -\frac{(r+3)^2}{(r+2)^2}a_1 = \frac{(r+3)^2}{(r+1)^2}a_0,\ a_3 = -\frac{(r+4)^2}{(r+3)^2}a_2 = -\frac{(r+4)^2}{(r+1)^2}a_0$$

and so on. Putting these values in (3.57), we get

$$y = a_0x^r[1 - \frac{(r+2)^2}{(r+1)^2}x + \frac{(r+3)^2}{(r+1)^2}x^2 - \frac{(r+4)^2}{(r+1)^2}x^3 + \cdots] \tag{3.58}$$

Differentiating partially equation (3.58) with respect to r, we get

$$\frac{\partial y}{\partial r} = a_0x^r logx\Big[1 - \frac{(r+2)^2}{(r+1)^2}x + \frac{(r+3)^2}{(r+1)^2}x^2 - \frac{(r+4)^2}{(r+1)^2}x^3 + \cdots\Big]$$
$$+a_0x^r\Big[-\frac{(r+2)^2}{(r+1)^2}x \times \frac{-2(r+2)}{(r+1)^3} + \frac{(r+3)^2x^2}{(r+1)^2} \times \frac{-2(r+3)}{(r+1)^2} - \frac{(r+4)^2x^3}{(r+1)^2} \times \frac{-2(r+2)}{(r+1)^3} + \cdots\Big]$$

Putting $r = 0$ and replacing a_0 by A in (3.58), we get

$$(y)_{r=0} = A\Big[1 - 2^2x^2 + 3^2x^4 - 4^2x^6 + \cdots\Big] = Au\,(say)$$

Next putting $r = 0$ in (3.59) and replacing a_0 by B, we get

$$\begin{aligned}(\frac{\partial y}{\partial r})_{r=0} &= B\ \log\ x\{1 - 2^2x^2 + 3^2x^4 - 4^2x^6 + \cdots\}\\ &+ B\Big[1 - 2^2x \times (-4) + 3^2x^2 \times (-6) - 4^2x^3 \times (-8) + \cdots\Big]\\ &= B\Big[u\ \log\ x + \Big\{1 + 16x - 54x^2 + 128x^3 + \cdots\Big\}\Big] = Bv\,(say)\end{aligned}$$

Hence the required solution is given by $y(x) = Au(x) + Bv(x),\ 0 < x < 1$, where A and B are arbitrary constants.

Example 3.17 Find the power series solution of the differential equation

$$y''(x) + xy'(x) + (x^2 + 2)y(x) = 0$$

about the point $x = 0$.

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 + 2)y = 0 \tag{3.59}$$

Comparing the above differential equation with (3.1) we have $p_1(x) = x$ and $p_0(x) = x^2 + 2$. Obviously, $x = 0$ is a ordinary point. Note that both $p_1(x)$ and $p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < \infty$. So let

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_nx^n,\ 0 < x < \infty \tag{3.60}$$

be the trial solution of (3.59). Differentiating twice in a succession (3.60) gives

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$$

Putting these values y, y' and y'' in (3.59) we get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + (x^2+2)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

We shift the index of summation in the first series by 2, replacing n with $n+2$ and using the initial value $n = 0$. We shift the index of summation in the third series by 2, replacing n with $n-2$ and using the initial value $n = 2$. Since we want to express everything in only one summation sign, we have to start the summation at $n = 2$ in every series,

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n + 2\sum_{n=1}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 + 2a_0 + (6a_3 + 3a_1)x + \sum_{n=1}^{\infty}\{(n+1)(n+2)a_{n+2} + (n+2)a_n + a_{n-2}\}x^n = 0$$

Equating the coefficients of various power of x to zero, we get

$$2a_2 = -2a_0 \Rightarrow a_2 = -a_0,\ 6a_3 + 3a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{2}$$

$$\text{and } (n+1)(n+2)a_{n+2} + (n+2)a_n + a_{n-2} = 0 \Rightarrow a_{n+2} = -\frac{(n+2)a_n + a_{n-2}}{(n+1)(n+2)}\ \forall n \geq 2$$

This last condition is called recurrence formula. Now putting $n = 2, 3, 4, \cdots$ successively in the above recurrence formula we get

$$a_4 = -\frac{4a_2 + a_0}{12} = -\frac{-4a_0 + a_0}{12} = -\frac{a_0}{4},\ a_5 = -\frac{5a_3 + a_1}{20} = -\frac{1}{4}a_3 - \frac{1}{20}a_1 = \frac{1}{8}a_1 - \frac{1}{20}a_1 = \frac{3a_1}{40}$$

and so on. Substituting the values of $a_2, a_3, a_4, \cdots$ in (3.60) the required solution is

$$\begin{aligned} y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ \Rightarrow y &= a_0 + a_1x - a_0x^2 - \frac{a_1}{2}x^3 - \frac{a_0}{4}x^4 + \frac{3a_1}{40}x^5 + \cdots \\ \Rightarrow y &= a_0(1 - x^2 - \frac{1}{4}x^4 + \cdots) + a_1(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 - \cdots),\ 0 < x < \infty \end{aligned}$$

Example 3.18 Determine the singular point of the following equation:

$$x(1-x)\frac{d^2y}{dx^2} + (3x-1)\frac{dy}{dx} + y = 0$$

Solution: Here the coefficient of $\frac{d^2y}{dx^2}$ is $x(1-x)$. Hence singular point is found by solving $x(1-x) = 0$ i.e. $x = 0, 1$. Hence 0 and 1 are the singular points.

Example 3.19 Find the ordinary point and singular points of

$$x^2(x+1)^2\frac{d^2y}{dx^2} + (x^2-1)\frac{dy}{dx} + 2xy = 0$$

Solution: Here the coefficient of $\frac{d^2y}{dx^2}$ is $x^2(1+x)^2$. Hence singular point is found by solving $x^2(1+x)^2 = 0$ i.e. $x = 0, -1$. Hence 0 and -1 are the singular points. For all values of x for which $x^2(1+x)^2 \neq 0$ are called ordinary points.

Example 3.20 Show that $x = 0$ ia regular point but $x = 2$ is not regular singular point of the equation

$$x(x-2)^3 y'' + 3(x-2)^3 y' + 4y = 0$$

Solution: Here the coefficient of $\frac{d^2y}{dx^2}$ is $x(x-2)^3$. Hence singular point is found by solving $x(x-2)^3 = 0$ i.e. $x = 0, 2$. Hence 0 and 2 are the singular points. Also, the given differential equation is compare with the differential equation (3.1) then $p_1(x) = \frac{3(x-2)^3}{x(x-2)^3}$ and $p_0(x) = \frac{4}{x(x-2)^3}$. Now since $\lim_{x\to x_0}(x-x_0)p_1(x) = \lim_{x\to 0}(x-0)\frac{3(x-2)^3}{x(x-2)^3} = 3$ and $\lim_{x\to x_0}(x-x_0)^2 p_0(x) = \lim_{x\to 0} x^2 \frac{4}{x(x-2)^3} = 0$ are both exist and finite so the point $x = 0$ is a regular singular point.
Again $\lim_{x\to x_0}(x-x_0)p_1(x) = \lim_{x\to 2}(x-2)\frac{3(x-2)^3}{x(x-2)^3} = 0$ and $\lim_{x\to x_0}(x-x_0)^2 p_0(x) = \lim_{x\to 2}(x-2)^2\frac{4}{x(x-2)^3} = \lim_{x\to 2}\frac{4}{x(x-2)}$ which does not exist. Hence $x = 2$ is not a regular singular point.

Example 3.21 Show that $x = 0$ is regular singular points but $x = -1$ is not regular singular point of

$$x(x+1)^3\frac{d^2y}{dx^2} + (x^2-1)\frac{dy}{dx} + 2y = 0$$

Solution: Here the coefficient of $\frac{d^2y}{dx^2}$ is $x(x+1)^3$. Hence singular point is found by solving $x(x+1)^3 = 0$ i.e. $x = 0, -1$. Hence 0 and -1 are the singular points. Also, the given differential equation is compare with the differential equation (3.1) then $p_1(x) = \frac{x^2-1}{x(x+1)^3}$ and $p_0(x) = \frac{2}{x(x+1)^3}$. Now since $\lim_{x\to x_0}(x-x_0)p_1(x) = \lim_{x\to 0}(x-0)\frac{x^2-1}{x(x+1)^3} = -1$ and $\lim_{x\to x_0}(x-x_0)^2 p_0(x) = \lim_{x\to 0}\frac{2x}{(x+1)^3} = 0$ are both exist and finite so the point $x = 0$ is a regular singular point.
Again $\lim_{x\to x_0}(x-x_0)p_1(x) = \lim_{x\to -1}(x+1)\frac{x^2-1}{x(x+1)^3} = \lim_{x\to -1}\frac{x-1}{x(x+1)}$ does not exist. Hence $x = -1$ is not a regular singular point.

Example 3.22 Show that $x = 0$ is the ordinary point

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 3y = 0.$$

Solution: The differential equation can be written as

$$\frac{d^2y}{dx^2} + \frac{x}{1+x^2}\frac{dy}{dx} - \frac{3}{1+x^2}y = 0.$$

Comparing the above differential equation with (3.1) we have $p_1(x) = \frac{x}{1+x^2}$ and $p_0(x) = -\frac{3}{1+x^2}$. Now since $\lim_{x\to x_0}(x-x_0)p_1(x) = \lim_{x\to 0}\frac{x^2}{(x^2+1)} = 0$ and $\lim_{x\to x_0}(x-x_0)^2 p_0(x) = \lim_{x\to 0} -\frac{3x^2}{(x^2+1)} = 0$. So $x = 0$ is an ordinary point of the given equation.

Example 3.23 Determine the singular points of the following differential equation and specify whether they are regular or irregular

$$(x-1)^4\frac{d^2y}{dx^2} + 2(x-1)\frac{dy}{dx} + y = 0$$

Solution: Here the coefficient of $\frac{d^2y}{dx^2}$ is $(x-1)^4$. Hence singular point is found by solving $(x-1)^4 = 0$ i.e. $x = 1$. Hence 1 is the singular point. Let the given differential equation is compare with the differential equation (3.1) then $p_1(x) = \frac{2(x-1)}{(x-1)^4}$ and $p_0(x) = \frac{1}{(x-1)^4}$. Now since $\lim_{x\to x_0}(x-x_0)p_1(x) = \lim_{x\to 1}(x-1)\frac{2(x-1)}{(x-1)^4} = \lim_{x\to 1}\frac{2}{(x-1)^2} = \infty$ is not finite so the point $x = 1$ is a irregular singular point.

Example 3.24 Assuming the solution of

$$(1 - x)y' + y = 0$$

has a series expansion about $x = 0$ work out the recursion relation. Write out the first few terms and show that the series terminates to give $y = A(1 - x)$ for arbitrary A.

Solution: The first order ODE can be written as

$$y' + \frac{y}{1 - x} = 0$$

hence $x = 0$ is the ordinary point. So we begin by writing

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ |x| < 1$$

and so by differentiation we get

$$y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}, \ |x| < 1 \tag{3.61}$$

and hence

$$xy'(x) = \sum_{n=0}^{\infty} a_n n x^n, \ |x| < 1.$$

Thus, substituting the differential equation we get

$$\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0, \ |x| < 1$$

In order to make progress we need to rewrite the first of these three series so that it is in the form

$$\sum_{n=0}^{\infty} \text{stuff}_n x^n$$

so that all three bits in the equation match. Well, let $m = n - 1$ in the expression for y', (3.61), to get

$$y'(x) = \sum_{m=0}^{\infty} a_{m+1}(m + 1)x^m. \tag{3.62}$$

In fact, this looks at first like it gives

$$y'(x) = \sum_{m=-1}^{\infty} a_{m+1}(m + 1)x^m \tag{3.63}$$

but the $m = -1$ term is zero, so that's fine. Now m is just an index so we can rename it n, don't get confused, this isn't the original n, we just want all parts of the equation to look the same.

In fact, we now have

$$\sum_{n=0}^{\infty} a_{n+1}(n + 1)x^n - \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0, \ |x| < 1$$

and we can group this all together to give

$$\sum_{n=0}^{\infty}[a_{n+1}(n+1)+(1-n)a_n]x^n = 0.$$

The recursion relation is

$$a_{n+1} = -\left(\frac{1-n}{1+n}\right)a_n$$

and this applies to n from zero upwards since that is what appears in the sum sign.

Starting at $n = 0$ we have

$$a_1 = -a_0.$$

For $n = 1$ we get

$$a_2 = 0$$

and the series terminates here because every term is something multiplied by the one before and so if a_2 is zero the rest of the series is zero. Thus $y = a_0(1-x)$ for arbitrary a_0.

Example 3.25 Assuming the solution of

$$y'' - 3x^2 y = 0$$

has a series expansion about $x = 0$ work out the recursion relation and write out the first four non-zero terms if $y(0) = 1$ and $y'(0) = 1$.

Solution: Since $x = 0$ is the ordinary point. Let the series solution of the above ODE be

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ |x| < \infty$$

This gives

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 3a_n x^{n+2} = 0, \ |x| < \infty$$

The problem here is with the powers of x. The easiest thing is to change everything to the highest power, in this case $n+2$. Hence, put $m+2 = n-2$ in the first sum

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=-4}^{\infty} (m+4)(m+3)a_{m+4}x^{m+2}$$

and substitute that back into the equation, writing m as n:

$$\sum_{n=-4}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2} - \sum_{n=0}^{\infty} 3a_n x^{n+2} = 0$$

and so the problem now is that the ranges are different. We need to take out the first few term of the first sum, well, the $n = -4$ and $n = -3$ terms are zero and so

$$\sum_{n=-4}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2} = 2a_2 + 6a_3 x + \sum_{n=0}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2}.$$

Now the equation reads

$$2a_2 + 6a_3x + \sum_{n=0}^{\infty}(n+4)(n+3)a_{n+4}x^{n+2} - \sum_{n=0}^{\infty}3a_nx^{n+2} = 0$$

$$\Rightarrow\ 2a_2 + 6a_3x + \sum_{n=0}^{\infty}\left[(n+4)(n+3)a_{n+4} - 3a_n\right]x^{n+2} = 0.$$

$$\text{Thus, } a_2 = 0,\ a_3 = 0,\ a_{n+4} = \frac{3}{(n+4)(n+3)}a_n$$

where the recursion relation applies for $n = 0, 1, \cdots$. Now, $y(0) = 1$ implies $a_0 = 1$ and $y'(0) = 1$ implies $a_1 = 1$, next, with $n = 0$, t he recursion gives

$$a_4 = \frac{1}{4}a_0 = \frac{1}{4} \text{ and with } n = 1,\ a_5 = \frac{3}{20}a_1 = \frac{3}{20}.$$

Now since $a_2 = a_3 = 0$ the $n = 2$ recursion gives $a_6 = 0$ and the $n = 3$ recursion gives $a_7 = 0$. However, $n = 4$ gives

$$a_8 = \frac{3}{32}a_4 = \frac{3}{128}$$

and so

$$y(x) = 1 + x + \frac{1}{4}x^4 + \frac{3}{20}x^5 + \frac{3}{128}x^8 + \cdots,\ |x| < \infty.$$

Aside. In the above we made all the powers the same as the highest power, this is usually the easiest thing, but it is just a matter of convenience. If we had decided to make them equal the smallest power instead, we would have substituted $n + 2 = m - 2$ in the second sum to get

$$\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=4}^{\infty}3a_{n-4}x^{n-2} = 0$$

and we would then remove the first four term from the first sum to get

$$2a_2 + 6a_3x + \sum_{n=4}^{\infty}\left[n(n-1)a_nx^{n-2} - 3a_{n-4}\right]x^{n-2} = 0$$

and so

$$a_2 = 0,\ a_3 = 0,\ a_n = \frac{3}{n(n-1)}a_{n-4}$$

where now the recursion relation applies to $n = 4, 5, \cdots$ because that is what is in the sum. Another way of proceeding is to define $a_{-4} = a_{-3} = a_{-2} = a_{-1} = 0$ and then rewrite the equation as

$$\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=0}^{\infty}3a_{n-4}x^{n-2} = 0$$

and carry on from there.

Example 3.26 Assuming the solution of

$$y' - 3xy = 2$$

has a series expansion about $x = 0$, work out the recursion relation and write out the first four non-zero terms.

Solution: Since $x = 0$ is the ordinary point of the above first order ODE. Also, the complication here is that unlike the other examples we have examined, this equation is an inhomogeneous equation. However, the thing to do is to press on with the same methods and hope for the best.

$$y = \sum_{n=0}^{\infty} a_n x^n, \ |x| < \infty$$

gives, when substituted into the equation,

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 3 a_n x^{n+1} = 2$$

and so the first problem is with the powers of x, let $m + 1 = n - 1$ in the first sum to give

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{m=-2}^{\infty} (m+2) a_{m+2} x^{m+1}$$

and, noting the the $m = -2$ term is zero, we take out the first two terms to get

$$a_1 + \sum_{n=0}^{\infty} [(n+2)a_{n+2} - 3a_n] x^{n+1} = 2.$$

Now, notice that the summand starts with an x term and so we get

$$a_1 = 2, \ a_{n+2} = \frac{3}{n+2} a_n.$$
$$\text{Thus, } a_2 = \frac{3}{2} a_0, \ a_3 = a_1 = 2 \text{ and so on.}$$

Hence, $y = a_0 \left(1 + \frac{3}{2}x^2 + \frac{3^2}{2 \times 4}x^4 + \frac{3^3}{2 \times 4 \times 6}x^6 + \cdots \right) + 2(x + x^3 + \frac{3}{5}x^5 + \frac{3^2}{5 \times 7}x^7 + \cdots), \ |x| < \infty$

and we see that the solution to this inhomogeneous solution has the usual structure: a particular part and a solution to the homogeneous equation depending on an arbitrary constant.

Example 3.27 Use the method of Froebenius to find series solutions for

$$xy'' + 2y' + xy = 0 \tag{3.64}$$

about $x = 0$.

Solution: The given differential equation can be written as

$$y''(x) + \frac{2}{x} y'(x) + y(x) = 0$$

Comparing the above differential equation with (3.1) we have $p_1(x) = \frac{2}{x}$ and $p_0(x) = 1$. Obviously, $x = 0$ is a singular point. Note that $xp_1(x) = 2$ and $x^2p_0(x) = x^2$. Both $xp_1(x)$ and $x^2p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < \infty$. Hence, $x = 0$ is a regular singular point. Let us assume that the trial solution of of equation (3.64) be

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r},\ a_0 \neq 0,\ 0 < x < \infty$$

Now, substituting into the equation gives

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 2(n+r)]a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0.$$

so, moving the first power up to the second one, this gives

$$\sum_{n=-2}^{\infty} [(n+2+r)(n+r+1) + 2(n+r+2)]a_{n+2} x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

or, taking the first two terms out

$$r(r+1)a_0 x^{r-1} + (r+1)(r+2)a_1 x^r + \sum_{n=0}^{\infty} [(n+2+r)(n+r+3)]a_{n+2} x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0.$$

So, if $r = 0$ or $r = -1$ then there is no constraint on a_0. Notice that $r = -1$ allows two solutions because, if $r = -1$ there is no equation for either a_0 or a_1. For $r = -1$ the recursion is

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}$$

so the first few non-zero terms are

$$y = \frac{1}{x}\left[a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right) + a_1\left(x - \frac{1}{6}x^3 \cdots\right)\right],\ 0 < x < \infty$$

For $r = 0$ the recursion is

$$a_{n+2} = -\frac{a_n}{(n+2)(n+3)}$$

and $a_1 = 0$, this means that the $r = 0$ solution is

$$y = a_0\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots\right),\ 0 < x < \infty$$

Notice that the $r = 0$ solution is actually just the a_1 solution for $r = -1$. This is just as well because there would be too many solutions otherwise. Notice the subtle way the method of Froebenius problems often work out. There is quite a lot to this subject we have only touched on. As an aside, notice the the solutions to the differential are $\cos x/x$ and $\sin x/x$. Writing these out as series will give the same thing as above.

Example 3.28 Assuming the solution of

$$(1 - x^2)y' - 2xy = 0$$

has a series expansion about $x = 0$, work out the recursion relation and write out the first four non-zero terms.

Solution: Assuming the solution of

$$(1-x^2)y' - 2xy = 0 \tag{3.65}$$

has a series expansion about $x = 0$, work out the recursion relation and write out the first four non-zero terms.

Solution: Since $x = 0$ is the ordinary point of the above differential equation(3.65). Let the series solution of the equation (3.65) near $x = 0$ be

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,\ |x| < 1 \text{ so } y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1},\ x^2 y'(x) = \sum_{n=0}^{\infty} a_n n x^{n+1},\ |x| < 1$$

and $xy = \sum_{n=0}^{\infty} a_n x^{n+1}$. Then the given equation becomes $\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n (n+2) x^{n+1} = 0.$

Once again, the first term is a problem because it doesn't have the same form as the other two. So, take

$$\sum_{n=0}^{\infty} a_n n x^{n-1}$$

and put $n-1 = m+1$ and, hence, $n = m+2$. When $n = 0$, $m = -2$ and when $n = 1$, $m = -1$. Thus

$$\sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{m=-2}^{\infty} a_{m+2}(m+2)x^{m+1}$$

and, once again renaming m as n we get

$$\sum_{n=-2}^{\infty} (n+2)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty} n a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

The problem now is with the range that the first sum runs over. The $n = -2$ term is no problem, it is zero, but the $n = -1$ term is a_1. Thus, we write

$$\sum_{n=-2}^{\infty} (n+2)a_{n+2}x^{n+1} = a_1 + \sum_{n=0}^{\infty} (n+2)a_{n+2}x^{n+1}$$

and the equation becomes

$$a_1 + \sum_{n=0}^{\infty} (n+2)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty} a_n n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Thus

$$a_1 + \sum_{n=0}^{\infty} [(n+2)a_{n+2} - n a_n - 2a_n]x^{n+1} = 0.$$

Notice that the summand starts with the x term. The recursion relation is therefore

$$a_{n+2} = a_n$$

with the additional conditions $a_1 = 0$. Hence, $a_6 = a_4 = a_2 = a_0$, $a_5 = a_3 = a_1 = 0$ and so on. The first four nonzero terms of the expansion gives

$$y = a_0(1 + x^2 + x^4 + x^6 + \cdots), \ |x| < 1.$$

Example 3.29 Assuming the solution of

$$y'' - 3y' + 2y = 0$$

has a series expansion about $x = 0$, by substitution, work out the recursion relation. If $y(0) = 1$ and $y'(0) = 0$ what are the first five non-zero terms.

Solution: Here $p_1(x) = -3$ and $p_0(x) = 2$. So $x = 0$ is the ordinary point. Let the series solution of the above ODE near $x = 0$ be

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ |x| < \infty$$

$$\text{so, } y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \ |x| < \infty \qquad \text{and} \qquad y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}, \ |x| < \infty$$

$$\text{Thus, } \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 3\sum_{n=0}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Again, we want to make each part look the same. As before, changing the index gives

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

The same thing can be done with the y'': let $m = n - 2$ to get

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=-2}^{\infty} (m+1)(m+2) a_{m+2} x^m$$

and the $m = -2$ and $m = -1$ terms are both zero, so, renaming the m as n we get

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 3\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

and this gives

$$\sum_{n=0}^{\infty} [(n+1)(n+2) a_{n+2} - 3(n+1) a_{n+1} + 2a_n] x^n = 0.$$

The recursion relation is

$$(n+1)(n+2) a_{n+2} - 3(n+1) a_{n+1} + 2a_n = 0.$$

Now apply the initial conditions, $y(0) = 1$ implies that $a_0 = 1$, $y'(0) = 0$ implies $a_1 = 0$. For $n = 0$ the recursion relation gives

$$2a_2 - 3a_1 + 2a_0 = 0$$

and so $a_2 = -a_0 = -1$. Next $n = 1$ gives

$$6a_3 - 6a_2 + 2a_1 = 0$$

and so $a_3 = a_2 = -a_0 = -1$. Next $n = 2$ gives

$$12a_4 - 9a_3 + 2a_2 = 0$$

and so $a_3 = a_2 = -a_0 = -1, a_4 = -\frac{7}{12}$. Next $n = 3$ gives

$$20a_5 - 12a_4 + 2a_3 = 0$$

and so $a_3 = a_2 = -a_0 = -1, a_4 = -\frac{7}{12}, a_5 = -\frac{1}{4}$. Therefore the first fives nonzero terms are

$$y(x) = 1 - x^2 - x^3 - \frac{7}{12}x^4 - \frac{1}{4}x^5 - \cdots, \; |x| < \infty.$$

3.12 Multiple Choice Questions

1. Consider the following statement P and Q: **GATE(MA)-2016**
   (P) : $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ has two linearly independent Frobenius series solution near $x = 0$.
   (Q) : $x^2y'' + 3\sin xy' + y = 0$ has two linearly independent Frobenius series solution near $x = 0$.
   which of the following statements hold TRUE?
   (A) both P and Q (B) only P (C) only Q (D) Neither P nor Q.
   Ans. (A).

2. Determine the singular points of the following differential equation

   $$x^2(x-1)^2\frac{d^2y}{dx^2} + 2(x-2)\frac{dy}{dx} + (x+3)y = 0$$

   A) $1, 3$ B) $-1, 0$ C) $0, 1$ D) $-1, -2$
   Ans. C)
   Hint. Let the given differential equation is compare with the differential equation (3.1) then $P_0(x) = x^2(x-1)^2, P_1(x) = 2(x-2)$ and $P_2(x) = (x+3)$. Since $P_0(0) = 0$ and $P_0(1) = 0$ so both 0 and 1 are the singular points.

3. If $\sum_{m=0}^{\infty} c_m x^{r+m}$ is assumed to be a solution of

   $$x^2y'' - xy' - 3(1+x^2)y = 0$$

   then the values of r are **GATE(MA)-12**
   A) $1, 3$ B) $-1, 3$ C) $1, -3$ D) $-1, -3$
   Ans. B)

4. For the differential equation

$$(x-1)y'' + (\cot \pi x)y' + (cosec^2 \pi x)y = 0,$$

which of the following statement is true **GATE(MA)-06**
(A) 0 is regular and 1 is irregular (B) 0 is regular and 1 is regular
(C) Both 0 and 1 are regular (D) Both 0 and 1 are irregular
Ans. A)

5. The initial value problem $xy'' + y' + xy = 0$, $y(0) = 0$, $(\frac{\partial y}{\partial x})_{x=0} = 0$ has **GATE(MA)-06**
(A) Unique solution (B) No solution
(C) Infinite number of Solution (D) Two independent solutions
Ans. B)

6. For the differential equation **GATE(MA)-05**

$$x^2(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

A) $x = 1$ is an ordinary point. B) $x = 1$ is a regular singular point.
C) $x = 0$ is an irregular singular point. D) $x = 0$ is an ordinary point.
Ans. B)

7. It is required to find the solution of differential equation

$$2x(2x+3)y'' + 2(3+x)y' - xy = 0$$

around $x = 0$. The roots of the indicial equation are **GATE(MA)-05**
A) $0, \frac{1}{2}$ B) $0, 2$ C) $\frac{1}{2}, \frac{1}{2}$ D) $0, -\frac{1}{2}$
Ans. D)

8. It is required to find the solution of differential equation

$$2x(2+x)y'' - 2(3+x)y' + xy = 0$$

around $x = 0$. The roots of the indicial equation are **GATE(MA)-05**
A) $0, \frac{1}{2}$ B) $0, 2$ C) $\frac{1}{2}, \frac{1}{2}$ D) $0, -\frac{1}{2}$
Ans. B)

9. The indicial equation for

$$x(1+x^2)y'' + (\cos x)y' + (1 - 3x + x^2)y = 0 \quad \text{is}$$

A) $r^2 - r = 0$ B) $r^2 + r = 0$ C) $r^2 = 0$ D) $r^2 - 1 = 0$ **GATE(MA)-04**
Ans. C)

10. For

$$x(x-1)y'' + (\sin x)y' + 2x(x-1)y = 0$$

consider the following statements
P: $x = 0$ is a regular singular point.
Q: $x = 1$ is a regular singular point. **GATE(MA)-08**
A) both P and Q are true. B) P is false and Q is true.
C) P is true and Q is false. D) both P and Q are false.
Ans. B)
Hint. $\frac{\sin x}{x} \to 0$ as $x \to 0$. So $x = 0$ is a ordinary point.

11. The ordinary differential equation

$$2x^2\frac{d^2y}{dx^2}+7x(x+1)\frac{dy}{dx}-3y=0.$$

Find the correct statement
A: $x=1$ is a regular singular point.
B: $x=1$ is a regular singular point.
C: all x($x\neq 0$) are ordinary points.
D: $x\neq 0$ is a regular singular point.
Ans. A) and C).
Hint.: The given differential equation $2x^2\frac{d^2y}{dx^2}+7x(x+1)\frac{dy}{dx}-3y=0$, can be written as $\frac{d^2y}{dx^2}+\frac{7x(x+1)}{2x^2}\frac{dy}{dx}-\frac{3}{2x^2}y=0$. The given differential equation is compare with the differential equation (3.1) then $p_1(x)=\frac{7x(x+1)}{2x^2}$ and $p_0(x)=-\frac{3}{2x^2}$. Since neither $\lim_{x\to 0}p_1(x)$ nor $\lim_{x\to 0}p_0(x)$ does exist. So $p_1(x)$ and $p_0(x)$ are not analytic at $x=0$. Hence, $x=0$ is the singular point of the said differential equation. Now $\lim_{x\to x_0}(x-x_0)p_1(x)=\lim_{x\to 0}(x-0)\frac{7x(x+1)}{2x^2}=\frac{7}{2}$ and $\lim_{x\to x_0}(x-x_0)^2p_0(x)=\lim_{x\to 0}(x-0)^2\left(\frac{-3}{2x^2}\right)=-\frac{3}{2}$ are both exist and finite so the point $x=0$ is a regular singular point. All points x $(x\neq 0)$ are ordinary points.

12. The ordinary differential equation

$$\frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{n(n+1)}{1-x^2}y=0 \tag{3.66}$$

Find the correct statement
A: $x=1$ is only a regular singular point.
B: $x=-1$ is only a regular singular point.
C: $x=\infty$ is only a singular point.
D: $x=1,-1,\infty$ are regular singular points.
Ans. D).
Hint.: Substituting $x=\frac{1}{t}$ to the given equation. Then

$$\frac{dy}{dx}=-t^2\frac{dy}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2}=t^4\frac{d^2y}{dt^2}+2t^3\frac{dy}{dt}$$

Using these substitution the given equation reduces to

$$t^4\frac{d^2y}{dt^2}+\frac{2t^3}{t^2-1}\frac{dy}{dt}+\frac{n(n+1)t^2}{t^2-1}y=0 \tag{3.67}$$

Since $t=0$ is a singular point of equation (3.67), so $x=\infty$ is a singular point of the given differential equation (3.66).
Similarly, we can show that $x=1,-1$ are also regular singular points of (3.66).

13. Suppose the equation

$$x^2y''-xy'+(1+x^2)y=0$$

has a solution of the form $y=\sum_{n=0}^{\infty}c_nx^{n+r}$. **GATE(MA)-07**
i) The indicial equation for r is
A) $r^2-1=0$ B) $(r-1)^2=0$ C) $(r+1)^2=0$ D) $r^2+1=0$

Ans. B)
ii) For $n \geq 2$ the co-efficient of c_n will be satisfy the relation
A) $n^2 c_n - c_{n-2} = 0$ B) $n^2 c_n + c_{n-2} = 0$
C) $c_n - c_{n-2} = 0$ D) $c_n + c_{n-2} = 0$
Ans. B)

14. If $y = \sum_{m=0}^{\infty} a_m x^m$ is a solution of $y'' + xy' + 3y = 0$ then $\frac{a_m}{a_{m+2}}$. **GATE(MA)-04**
A) $\frac{(m+1)(m+2)}{m+3}$ B) $-\frac{(m+1)(m+2)}{m+3}$ C) $-\frac{m(m-1)}{m+3}$ D) $\frac{m(m-1)}{m+3}$
Ans. B)

3.13 Review Exercise

1 Show that $x = 0$ is an ordinary point of $(x^2 + 1)y'' + xy' - xy = 0$.

2 Show that $y'' + e^x y = 0$ has a solution ϕ of the form

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$$

which satisfies $\phi(0) = 1,\ \phi'(0) = 0$.

3 Determine all the singular points of the equation $2x^2 y'' - xy' + (x + 2)y = 0$.
Ans: $x = 0$.

4 Show that $x = 0$ is a regular singular point and $x = -1$ is not regular singular points of $x^2(x + 1)^2 y'' + (x^2 - 1)y' + 2y = 0$.

5 Show that infinity is not a regular singular point for the equation

$$y'' + ay' + by = 0,$$

where a, b are constants, not both zero.

6 Show that infinity is not a regular singular point for the Bessel equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0.$$

Hint. See the section 5.2 in Chapter 5.

7 Show that infinity is a regular singular point for the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is constant.

8 Find the power series solution of of the following equations
(i) $(1 - x^2)y'' + 2xy' - y = 0$ about $x = 0$.
[**Ans** :$y = A(1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 - \cdots) + B(x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \cdots),\ |x| < 1$]
(ii)$(1 + x^2)y'' + xy' - y = 0$ about $x = 0$.
[**Ans** :$y = A(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{15}x^6 - \cdots) + Bx,\ |x| < 1$]
(iii)$(x^2 - 1)y'' + 4xy' + 2y = 0$ about $x = 0$.

[**Ans** :$y = A(1 + x^2 + x^4 + \cdots) + B(x + x^3 + x^5 + \cdots),\ |x| < 1$]
(iv)$y'' - xy' + 2y = 0$ about $x = 1$.
[**Ans** :$y = A(1 - (x-1)^2 - \frac{1}{3}(x-1)^3 - \cdots) + B((x-1) + \frac{1}{2}(x-1)^2 - \cdots),\ |x-1| < 1$].

9 Find the power series solution of the initial value problem
(i) $(1 - x^2)y'' + 2y = 0$, $y(0) = 4$, $y'(0) = 5$.
[**Ans** :$y = 4 + 5x - 4x^2 - \frac{5}{3}x^3 - \frac{1}{3}x^5 + \cdots,\ -1 < x < 1$]
(ii) $(x^2 - 1)y'' + 3xy' + xy = 0$, $y(0) = 2$, $y'(0) = 3$.
[**Ans**.$y = 2 + 3x + \frac{11}{6}x^3 + \frac{1}{4}x^4 - \cdots,\ -1 < x < 1$]
(iii)$(x^2 - 1)y'' + 3xy' + xy = 0$, $y(2) = 4$, $y'(2) = 6$.
[**Ans** :$y = 4 + 6(x-2) - \frac{22}{3}(x-2)^2 + \frac{169}{27}(x-2)^3 - \cdots$.

10 Solve $xy'' + (1 + x)y' + 2y = 0$ in series near $x = 0$.
[**Ans** :$y = A(1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 + \cdots) + B\log x\{1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 + 2(2 - \frac{1}{2})x - \frac{3}{2!}(-\frac{1}{3} + 2 + \frac{1}{2})x^2 + \cdots\},\ 0 < x < \infty$]

11 Solve $2x^2y'' + xy' - (x + 1)y = 0$ in series near $x = 0$.
[**Ans** :$y = Ax(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \cdots) + Bx^{-\frac{1}{2}}(1 - x - \frac{1}{2}x^2 + \cdots),\ 0 < x < \infty$]

12 Solve $x^2(x + 1)y'' + x(x + 1)y' - y = 0$
[**Ans** :$y = A(1 - \frac{x}{3} + \frac{x^2}{6} + \cdots) + Bx^{-1}(1 + x),\ 0 < x < 1$]

13 Solve $x^2y'' + y' + y = 0$
[**Ans** :$y = (A + B\log x)(1 - \frac{x}{1} + \frac{x^2}{1^2.2^2} + \cdots) + 2B(1 - \frac{3x^2}{2} + \cdots),\ 0 < x < \infty$]

14 Solve $2x^2(x - 1)y'' + x(3x + 1)y' - 2y = 0$ in series, convergent near $x = \infty$.
Ans: $y = A\left[1 + \frac{2}{x} + \frac{7}{3x^2} + \cdots\right] + Bx^{\frac{-1}{2}}\left[1 + \frac{4}{3x} + \frac{22}{15x^2} + \cdots\right]$ in $|x| > 1$.
Hint. Put $x = \frac{1}{z}$, we get $\frac{dy}{dx} = -z^2\frac{dy}{dz}$ and $\frac{d^2y}{dx^2} = z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz}$. Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get $2(z - z^2)\frac{d^2y}{dz^2} + (1 - 5z)\frac{dy}{dz} - 2y = 0$ which is to be solve in the series about $z = 0$ is a regular singular point. Then the solution is $y = A\left[1 + \frac{2}{x} + \frac{7}{3x^2} + \cdots\right] + Bx^{\frac{-1}{2}}\left[1 + \frac{4}{3x} + \frac{22}{15x^2} + \cdots\right]$ in $|x| > 1$.

15 Solve $y'' + x^2y = 2 + x + x^2$ about $x = 0$.
Ans: $y = A\left(1 - \frac{x^4}{12} + \frac{x^8}{672} - \cdots\right) + B\left(x - \frac{x^5}{20} + \frac{x^9}{140} - \cdots\right) + x^2 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{30} + \cdots$.

16 Solve $y'' - y = x$ in power of x.
Ans: $y = A\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) + B\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$.

Chapter 4

Legendre Equations

4.1 Introduction

Legendres equation occurs in many areas of applied mathematics, physics, biomathematics and chemistry in physical situations. These polynomials may be constructed as a consequence of demanding a complete, orthogonal set of functions over the interval $[-1, 1]$. In quantum mechanics they represent angular momentum eigenfunctions. The Legendre functions are the solutions of **Legendres equation**, a second order linear differential equation with variable coefficients. The said equation was introduced by Legendre in the late 18^{th} century.

4.2 Legendre's Equation

The ordinary differential equation of the from

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \tag{4.1}$$

$$\Rightarrow \quad \frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

is called Legendre's equation where n is a positive integer, called the order of the equation. Now, comparing the above differential equation with (3.1), we have $p_0(x) = \frac{n(n+1)}{1-x^2}$, $p_1(x) = -\frac{2x}{1-x^2}$. So $x = \pm 1$ is the singular points. Next, $p_0(0) = n(n+1)$, $p_1(0) = 0$ so the point $x = 0$ is the ordinary point. Also $\lim_{x\to 1}(x-1)p_1(x) = 1$ and $\lim_{x\to 1}(x-1)^2 p_0(x) = 0$, so $x = 1$ is a regular singular point of (4.1). Similarly, $x = -1$ is also a regular singular point of the said differential equation. The physically interesting range for x is $-1 < x < 1$. Also from the previous example-3.4, we say that $x = \infty$ is singular point of Legendres equation (4.1). We solve (4.1) in series of descending powers of x. Let the series solution of (4.1) is

$$y = \sum_{m=0}^{\infty} a_m x^{r-m}; a_0 \neq 0, \ -1 < x < 1 \tag{4.2}$$

Differentiating twice (4.2) in a succession, we get

$$y' = \sum_{m=0}^{\infty}(r-m)a_m x^{r-m-1} \text{ and } y'' = \sum_{m=0}^{\infty}(r-m)(r-m-1)a_m x^{r-m-2}$$

Putting these values y, $y^{'}$ and $y^{''}$ in (4.1) we get

$$(1-x^2)\sum_{m=0}^{\infty}(r-m)(r-m-1)a_m x^{r-m-2} - 2x\sum_{m=0}^{\infty}(r-m)a_m x^{r-m-1} + n(n+1)\sum_{m=0}^{\infty}a_m x^{r-m} = 0$$
$$\Rightarrow \sum_{m=0}^{\infty}(r-m)(r-m-1)a_m x^{r-m-2} - \sum_{m=0}^{\infty}(r-m)(r-m-1)a_m x^{r-m}$$
$$-2\sum_{m=0}^{\infty}(r-m)a_m x^{r-m} + n(n+1)\sum_{m=0}^{\infty}a_m x^{r-m} = 0$$
$$\Rightarrow \sum_{m=0}^{\infty}(r-m)(r-m-1)a_m x^{r-m-2} - \sum_{m=0}^{\infty}a_m(r-m-n)(r-m+n+1)x^{r-m} = 0$$

Equating to zero the coefficient of highest power of x, namely x^r,we obtain

$$a_0(r-n)(r+n+1) = 0$$

Since $a_0 \neq 0$ we have the indicial equation as

$(r-n)(r+n+1) = 0$ so that the roots of the indicial equation are $r = n$ and $r = -(n+1)$.

Here the roots of the indicial equation are unequal and differ by an integer. Next equating zero the coefficient of x^{r-m}, we obtain the recurrence relation as

$$(r-m+2)(r-m+1)a_{m-2} - (r-m-n)(r-m+n+1)a_m = 0$$
$$\Rightarrow \quad a_m = \frac{(r-m+2)(r-m+1)}{(r-m-n)(r-m+n+1)}a_{m-2} \tag{4.3}$$

Next equating the coefficient x^{r-1} and we get $a_1(r-1-n)(r+n) = 0$. As $r = n$ and $r = -(n+1)$, so neither $(r-1-n)$ nor $(r+n)$ is zero. Therefore $a_1 = 0$ for both the roots of indicial equation $r = n$ and $r = -(n+1)$. Using $a_1 = 0$ and (4.3), we get

$$a_1 = a_3 = a_5 = a_7 = \cdots = 0$$

To obtain $a_2, a_4, a_6, a_8, \cdots$, we consider two cases
Case-I: When $r = n$. Then (4.3) becomes

$$a_m = -\frac{(n-m+2)(n-m+1)}{m(2n-m+1)}a_{m-2} \tag{4.4}$$

Putting $m = 2, 4, 6, \cdots$ in (4.4), we have

$$a_2 = -\frac{n(n-1)}{2(2n-1)}a_0$$
$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)}a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}a_0 \text{ and so on}$$

Putting these values in (4.2), and replacing a_0 by A, we get

$$y = A\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \cdots\right] \tag{4.5}$$

The last term is constant when n is even or contain x when n is odd.
Case-II: When $r = -(n+1)$. Then (4.3) becomes

$$a_m = \frac{(n+m-1)(n+m)}{m(2n+m+1)} a_{m-2} \tag{4.6}$$

Putting $m = 2, 4, 6, \cdots$ in (4.6), we have

$$\begin{aligned} a_2 &= \frac{(n+1)(n+2)}{2(2n+3)} a_0 \\ a_4 &= \frac{(n+3)(n+4)}{4(2n+5)} a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} a_0 \text{ and so on} \end{aligned}$$

Putting these values in (4.2), and replacing a_0 by B, we get

$$y = B\Big[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \cdots\Big] \tag{4.7}$$

Thus, the two independent solutions of Legendre's equation are given by equation (4.5) and (4.7). If we take $a = \frac{[1.3.5\cdots(2n-1)]}{n!}$, the solution (4.5) is called **Legendre's function of the first kind or Legendre's polynomial of degree** n **and is denoted by** $P_n(x)$. If $b = \frac{n}{[1.3.5\cdots(2n+1)]}$, the solution (4.7) is called **Legendre's function of the second kind and is denoted by** $Q_n(x)$. Hence the general solution of (4.1) is given by $y = AP_n(x) + BQ_n(x)$, where A and B are arbitrary constants.

4.3 Legendre's polynomial of degree n

Legendre's function or polynomial of degree n is denoted by $P_n(x)$ and is defined by

$$\begin{aligned} P_n(x) &= \frac{[1.3.5\cdots(2n-1)]}{n!}\Big[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} - \cdots\Big] \\ \Rightarrow P_n(x) &= \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}, \qquad \text{where } \Big[\frac{n}{2}\Big] = \begin{cases} \frac{n}{2}\,; \text{if } n \text{ is even} \\ \frac{n-1}{2}\,; \text{if } n \text{ is odd} \end{cases} \end{aligned} \tag{4.8}$$

Putting $n = 0, 1, 2, 3, 4, 5, \cdots$ in (4.8), we get

$$\begin{aligned} P_0(x) &= \frac{1}{0!} x^0 = 1 \\ P_1(x) &= \frac{1}{1!} x^1 = x \\ P_2(x) &= \frac{1.3}{2!}\Big[x^2 - \frac{2.1}{2.3}\Big] = \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1.3.5}{3!}\Big[x^3 - \frac{3.2}{2.5} x\Big] = \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1.3.5.7}{4!}\Big[x^4 - \frac{4.3}{2.7} x^2 + \frac{4.3.2.1}{2.4.7.5}\Big] = \frac{1}{8}(35x^4 - 30x^2 + 3) \\ P_5(x) &= \frac{1.3.5.7.9}{5!}\Big[x^5 - \frac{5.4}{2.9} x^3 + \frac{5.4.3.2}{2.4.9.7} x\Big] = \frac{1}{8}(63x^5 - 70x^3 + 15x) \text{ and so on} \end{aligned}$$

4.4 Generating Function For Legendre's Polynomial

Theorem 4.1 $P_n(x)$ is the coefficient of h^n in the expansion of $(1-2xh+h^2)^{-\frac{1}{2}}$ in the ascending power of x, i.e

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x),\ |x| \le 1,\ |h| < 1$$

Proof: Since $|h| < 1$ and $|x| \le 1$, we get

$$\begin{aligned}(1-2xh+h^2)^{-\frac{1}{2}} &= [1-h(2x-h)]^{-\frac{1}{2}} = 1 + \frac{1}{2}h(2x-h) + \frac{1.3}{2.4}h^2(2x-h)^2 + \cdots \\ &+ \frac{1.3\cdots(2n-3)}{2.4\cdots(2n-2)}h^{n-1}(2x-h)^{n-1} + \frac{1.3\cdots(2n-3)(2n-1)}{2.4\cdots(2n-2)2n}h^n(2x-h)^n + \cdots \quad (4.9)\end{aligned}$$

Now the coefficient of h^n in $\frac{1.3\cdots(2n-3)(2n-1)}{2\cdot4\cdots(2n-2)2n}h^n(2x-h)^n$ is

$$\begin{aligned}&= \frac{1.3\cdots(2n-3)(2n-1)}{2.4\cdots(2n-2)2n}(2x)^n = \frac{1.3\cdots(2n-1)2^n}{(2\cdot1)\cdot(2\cdot2)\cdot(2\cdot3)\cdots(2\cdot n)}x^n \\ &= \frac{1.3\cdots(2n-1)}{2^n\cdot n!}2^n x^n = \frac{1\cdot3\cdots(2n-1)}{n!}x^n \quad (4.10)\end{aligned}$$

Again coefficient of h^n in $\frac{1.3\cdots(2n-3)}{2\cdot4\cdots(2n-2)}h^{n-1}(2x-h)^{n-1}$ is

$$\begin{aligned}&= \frac{1\cdot3\cdots(2n-3)}{(2\cdot1)\cdot(2\cdot2)\cdots(2(n-1))}\{-(n-1)(2x)^{n-2}\} \\ &= -\frac{1\cdot3\cdots(2n-3)}{2^{n-1}\cdot1\cdot2\cdot3\cdots(n-1)}\cdot\frac{2n-1}{n}\cdot\frac{n}{2n-1}\{(n-1)\cdot2^{n-2}\cdot x^{n-2}\} \\ &= -\frac{1\cdot3\cdots(2n-3)(2n-1)}{n!}\frac{n(n-1)}{2(2n-1)}x^{n-2} \text{ and so on.} \quad (4.11)\end{aligned}$$

Using (4.10), (4.11)$\cdots$ we see that the coefficient of h^n in the expansion of $(1-2xh+h^2)^{-\frac{1}{2}}$ is given by

$$\frac{[1\cdot3\cdot5\cdots(2n-1)]}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right] = P_n(x)$$

Now $P_1(x), P_2(x),\cdots$ will be the coefficient of $h, h^2,\cdots$ in the expansion of $(1-2xh+h^2)^{-\frac{1}{2}}$.

$$\text{Therefore } (1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x).$$

Theorem 4.2 Show that (a) $P_n(1) = 1$ (b) $P_n(-x) = (-1)^n P_n(x)$, hence deduce that $P_n(-1) = (-1)^n$ where $P_n(x)$ is the Legendre's Polynomials of degree n.

Proof: (a) From the generating function, we know that

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Putting $x = 1$, we get

$$(1-2h+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(1) \Rightarrow \sum_{n=0}^{\infty} h^n P_n(1) = (1-h)^{-1} = 1 + h + h^2 + \cdots = \sum_{n=0}^{\infty} h^n$$

Equating the coefficient of h^n, we get $P_n(1) = 1$.
(b) From the generating function, we know that

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{4.12}$$

Replacing h by $-h$ in (4.12)

$$(1+2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-h)^n P_n(x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x) \tag{4.13}$$

Again replacing x by $-x$ in (4.12) we get,

$$(1+2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(-x) \tag{4.14}$$

From (4.13) and (4.14), we get

$$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x)$$

Equating the coefficient of h^n, we get

$$P_n(-x) = (-1)^n P_n(x)$$

Next putting $x = 1$, we get

$$P_n(-1) = (-1)^n P_n(1) \Rightarrow P_n(-1) = (-1)^n, \text{ [Since } P_n(1) = 1]$$

Theorem 4.3 (Recurrence Relation I) Prove that

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), n \geq 1 \text{ or}$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x), n \geq 2$$

where $P_n(x)$ is the Legendre's Polynomial of degree n.

Proof: From the generating function, we get

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x),\ |x| \leq 1,\ |h| < 1 \tag{4.15}$$

Differentiating both side of (4.15) with respect to h, we get

$$-\frac{1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1} P_n(x) \tag{4.16}$$

Multiplying both sides by $(1-2xh+h^2)$, (4.16) gives

$$(x-h)(1-2xh+h^2)^{-\frac{1}{2}} = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

$$\Rightarrow \quad (x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{n=1}^{\infty} nh^{n-1}P_n(x) \text{[By (4.15)]}$$

$$\Rightarrow \quad x\sum_{n=0}^{\infty} h^n P_n(x) - \sum_{n=0}^{\infty} h^{n+1}P_n(x) = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) - 2x\sum_{n=0}^{\infty} nh^n P_n(x) + \sum_{n=0}^{\infty} nh^{n+1}P_n(x)$$

Equating the coefficient of h^n from both sides, we get

$$xp_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Replacing n by $(n-1)$, we get, $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$. (4.17)

Theorem 4.4 (Recurrence Relation II) Prove that

$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$, where $P_n(x)$ is the Legendre's Polynomial of degree n.

Proof: From the generating function, we get

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x), \ |x| \le 1, \ |h| < 1 \tag{4.18}$$

Differentiating both side of (4.18) with respect to h, we get

$$-\frac{1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) \tag{4.19}$$

Again, differentiating (4.18) with respect to x, we get,

$$h(1-2xh+h^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} h^n P_n'(x)$$

$$\Rightarrow \quad h(x-h)(1-2xh+h^2)^{-\frac{3}{2}} = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \text{ [Multiplying both side by } (x-h)]$$

$$\Rightarrow \quad h\sum_{n=0}^{\infty} nh^{n-1}P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \text{ [by (4.19)]}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} nh^n P_n(x) = x\sum_{n=0}^{\infty} h^n P_n'(x) - \sum_{n=0}^{\infty} h^{n+1}P_n'(x)$$

Equating the coefficient of h^n on both sides we get $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$.

Theorem 4.5 (Recurrence Relation III) Prove that

$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$, where $P_n(x)$ is the Legendre's Polynomial of degree n.

Proof: From recurrence relation-I, we get

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Differentiating with respect to x, we get

$$\begin{aligned}
& (2n+1)xP_n'(x) + (2n+1)P_n(x) = (n+1)P_{n+1}'(x) + nP_{n-1}'(x) \\
\Rightarrow\ & (2n+1)(nP_n(x) + P_{n-1}'(x)) + (2n+1)P_n(x) = (n+1)P_{n+1}'(x) + nP_{n-1}'(x) \\
& [\text{from recurrence relation-II, } \ xP_n'(x) = nP_n(x) + P_{n-1}'(x)] \\
\Rightarrow\ & (2n+1)(n+1)P_n(x) = (n+1)P_{n+1}'(x) - (n+1)P_{n-1}'(x) \\
\Rightarrow\ & (2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)
\end{aligned}$$

Theorem 4.6 (Recurrence Relation IV) Prove that

$(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$, where $P_n(x)$ is the Legendre's Polynomial of degree n.

Proof: From recurrence relations II and III, we get

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x) \tag{4.20}$$

and $$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \tag{4.21}$$

Subtracting (4.20) from (4.21), we get, $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$.

Theorem 4.7 (Recurrence Relation V) Prove that

$(1-x^2)P_n'(x) = n(P_{n-1}(x) - xP_n(x))$, where $P_n(x)$ is the Legendre's Polynomial of degree n.

Proof: From recurrence relations II and IV, we get

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x) \tag{4.22}$$

and $$(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x) \tag{4.23}$$

Replacing n by $(n-1)$ in (4.23)

$$nP_{n-1}(x) = P_n'(x) - xP_{n-1}'(x) \tag{4.24}$$

Multiplying both side of (4.22) by x , we get

$$nxP_n(x) = x^2P_n'(x) - xP_{n-1}'(x) \tag{4.25}$$

Subtracting (4.25) from (4.24) , we get, $n(P_{n-1}(x) - xP_n(x)) = (1-x^2)P_n'(x)$.

Theorem 4.8 (Recurrence Relation VI) Prove that

$(1-x^2)P_n'(x) = (n+1)(xP_n(x) - P_{n+1}(x))$, where $P_n(x)$ is the Legendre's Polynomial of degree n.

Proof: From recurrence relation I , we get

$$\begin{aligned}
& (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \\
\Rightarrow\ & [(n+1)+n]xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \\
\Rightarrow\ & (n+1)(xP_n(x) - P_{n+1}(x)) = n(P_{n-1}(x) - xP_n(x)) \qquad (4.26)
\end{aligned}$$

From recurrence relation V, we get

$$(1-x^2)P_n'(x) = n(P_{n-1}(x) - xP_n(x)) \tag{4.27}$$

From equation (4.26) and (4.27), we get

$$(1-x^2)P'_n(x) = (n+1)(xP_n(x) - P_{n+1}(x))$$

4.5 Orthogonal Properties of Legendre's Function

Theorem 4.9 Prove that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0 \text{ if } m \neq n.$$

Where $P_n(x)$ and $P_m(x)$ are the Legendre's polynomial of degree n and m respectively.

Proof: The Legendre's equation is given by

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Since $P_n(x)$ and $P_m(x)$ satisfies Legendre's equation, we get

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0 \tag{4.28}$$
$$(1-x^2)P''_m(x) - 2xP'_m(x) + m(m+1)P_m(x) = 0 \tag{4.29}$$

Multiplying (4.28) by $P_m(x)$ and (4.29) by $P_n(x)$ and then subtracting the resulting equation, we get

$$\begin{aligned}
& (1-x^2)(P_n(x)P''_m(x) - P_m(x)P''_n(x)) - 2x(P_n(x)P'_m(x) - P_m(x)P'_n(x)) + \\
& [(m(m+1) - n(n+1))]P_m(x)P_n(x) = 0 \\
\Rightarrow\quad & (1-x^2)\frac{d}{dx}\Big\{P_n(x)P'_m(x) - P_m(x)P'_n(x)\Big\} - 2x\Big\{P_n(x)P'_m(x) - P_m(x)P'_n(x)\Big\} \\
& = [(n-m)(n+m+1)]P_m(x)P_n(x) \\
\Rightarrow\quad & \frac{d}{dx}\Big\{(1-x^2)(P_n(x)P'_m(x) - P_m(x)P'_n(x))\Big\} = [(n-m)(n+m+1)]P_m(x)P_n(x)
\end{aligned}$$

Integrating both side with respect to x from -1 to 1, we get

$$\int_{-1}^{1} \frac{d}{dx}\{(1-x^2)(P_n(x)P'_m(x) - P_m(x)P'_n(x))\}dx = [(n-m)(n+m+1)]\int_{-1}^{1} P_m(x)P_n(x)dx$$

$$\Rightarrow\quad [(1-x^2)(P_n(x)P'_m(x) - P_m(x)P'_n(x))]_{-1}^{1} = [(n-m)(n+m+1)]\int_{-1}^{1} P_m(x)P_n(x)dx$$

$$\text{Therefore}\quad \int_{-1}^{1} P_m(x)P_n(x)dx = 0, \text{ as } m \neq n.$$

Theorem 4.10 Prove that

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}, \text{ where } P_n(x) \text{ is the Legendre's polynomial of degree n.}$$

Proof: From generating function of Legendre polynomials, we get

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-\frac{1}{2}}, |x| \le 1, |h| < 1 \tag{4.30}$$

or
$$\sum_{m=0}^{\infty} h^m P_m(x) = (1-2xh+h^2)^{-\frac{1}{2}}, |x| \le 1, |h| < 1 \tag{4.31}$$

Multiplying the corresponding sides of (4.30) and (4.31), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h^n h^m P_n(x)P_m(x) = (1-2xh+h^2)^{-1}$$

Integrating both sides of the above equation with respect to x we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-1}^{1}\{P_n(x)P_m(x)dx\}h^{n+m} = \int_{-1}^{1}(1-2xh+h^2)^{-1}dx \tag{4.32}$$

$$\text{As,} \int_{-1}^{1} P_m(x)P_n(x)dx = 0 \quad m \neq n \tag{4.33}$$

Then (4.32) reduces to

$$\sum_{n=0}^{\infty}\int_{-1}^{1}[\{P_n(x)\}^2dx]h^{2n} = \int_{-1}^{1}(1-2xh+h^2)^{-1}dx$$

$$\Rightarrow \sum_{n=0}^{\infty}\int_{-1}^{1}[\{P_n(x)\}^2dx]h^{2n} = \left[\frac{\log(1+h^2-2hx)}{-2h}\right]_{-1}^{1}$$

$$\Rightarrow \sum_{n=0}^{\infty}\int_{-1}^{1}[\{P_n(x)\}^2dx]h^{2n} = -\frac{1}{2h}\Big[\log(1-h)^2 - \log(1+h)^2\Big]$$

$$\Rightarrow \sum_{n=0}^{\infty}\int_{-1}^{1}[\{P_n(x)\}^2dx]h^{2n} = \frac{1}{h}\Big[\log(1+h) - \log(1-h)\Big]$$

$$\Rightarrow \sum_{n=0}^{\infty}\int_{-1}^{1}[\{P_n(x)\}^2dx]h^{2n} = \frac{1}{h}\Big[(h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots) - (-h - \frac{h^2}{2} - \frac{h^3}{3} - \cdots)\Big]$$

$$\Rightarrow \sum_{n=0}^{\infty}\int_{-1}^{1}[\{P_n(x)\}^2dx]h^{2n} = \frac{2}{h}\Big[h + \frac{h^3}{3} + \frac{h^5}{5} + \cdots\Big]$$

$$\Rightarrow \sum_{n=0}^{\infty}\int_{-1}^{1}\Big[\{P_n(x)\}^2dx\Big]h^{2n} = \frac{2}{h}\sum_{n=0}^{\infty}\frac{h^{2n+1}}{2n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty}\int_{-1}^{1}[\{P_n(x)\}^2dx]h^{2n} = \sum_{n=0}^{\infty}\frac{2h^{2n}}{2n+1}$$

Equating the coefficient of h^{2n} both sides, we get, $\int_{-1}^{1} \left\{P_n(x)\right\}^2 dx = \frac{2}{2n+1}$.

Theorem 4.11 Prove that

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left\{(x^2-1)^n\right\}$$

where $P_n(x)$ is the Legendre's Polynomial of degree n which is also called Rodrigue's formula.

Proof: Let

$$y = (x^2-1)^n$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = n(x^2-1)^{n-1} \cdot 2x$$

Multiplying both side by (x^2-1), we get

$$(x^2-1)\frac{dy}{dx} = 2nx(x^2-1)^n \Rightarrow (x^2-1)\frac{dy}{dx} = 2nxy$$

Differentiating with respect to x, $(n+1)$ times by using the Leibnitz's theorem, we get

$$\begin{aligned} & (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + (n+1)\frac{d^{n+1}y}{dx^{n+1}} \cdot 2x + \frac{n(n+1)}{2}\frac{d^n y}{dx^n} \cdot 2 = 2n[x\frac{d^{n+1}y}{dx^{n+1}} + (n+1)\frac{d^n y}{dx^n}] \\ \Rightarrow \quad & (1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^n y}{dx^n} = 0 \end{aligned}$$

Putting $z = \frac{d^n y}{dx^n}$, we get, $(1-x^2)\frac{d^2 z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0$ which is the Legendre's equation and one of the solution is

$$z = cP_n(x) \Rightarrow \frac{d^n y}{dx^n} = cP_n(x) \tag{4.34}$$

Putting $x = 1$, we get, $c = [\frac{d^n y}{dx^n}]_{x=1}$, since $P_n(1) = 1$. Now $y = (x^2-1)^n = (x-1)^n(x+1)^n$. Differentiating n times with respect to x using Leibnitz's theorem, we get $\frac{d^n y}{dx^n} = (x^2-1)^n.n! + {}^nC_1 \frac{n!}{1!}.(x+1)n.(x-1)^{n-1} + \cdots + (x+1)^n.n!$. Therefore $[\frac{d^n y}{dx^n}]_{x=1} = (1+1)^n.n! = 2^n.n!$. Hence $c = [\frac{d^n y}{dx^n}]_{x=1} = 2^n \cdot n!$. Therefore from (4.34), we get, $P_n(x) = \frac{1}{c}\frac{d^n y}{dx^n} = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n$.

4.6 Expansion of $f(x)$ in a series of Legendre Polynomials

Supposing the expansion of $f(x)$ in a series Legendre polynomials to be possible, we write

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) \tag{4.35}$$

where $a_0, a_1, a_2, \cdots, a_n$ are constants. Multiplying both side of (4.35) by $P_n(x)$ and then integrating both side with respect to $'x'$ from -1 to 1, we get

$$\int_{-1}^{1} f(x)P_n(x)dx = \sum_{m=0}^{\infty}\left\{\int_{-1}^{1} a_m P_m(x)P_n(x)dx\right\}$$

$$\Rightarrow \int_{-1}^{1} f(x)P_n(x)dx = a_n\frac{2}{2n+1} \quad \left[as \int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0; \text{ if } m \neq n \\ \frac{2}{2n+1}; \text{ if } m = n \end{cases}\right]$$

$$\Rightarrow a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)dx$$

Example 4.1 If

$$f(x) = \begin{cases} 0; & \text{where } -1 < x < 0 \\ x; & \text{where } 0 < x < 1 \end{cases}$$

Show that

$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \cdots$$

Solution: Given that $f(x) = \begin{cases} 0 & ; \text{ where } -1 < x < 0 \\ x & ; \text{ where } 0 < x < 1 \end{cases}$. We know that

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{4.36}$$

$$\text{where } a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)dx = \left(n+\frac{1}{2}\right)\int_{0}^{1} xP_n(x)dx, \ \because f(x) = 0, \ -1 < x < 0. \tag{4.37}$$

Putting $n = 0, 1, 2, \cdots$ successively in (4.37), we get

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^1 xP_0(x)dx = \frac{1}{2}\int_0^1 xdx = \frac{1}{4}, \ a_1 = \frac{3}{2}\int_0^1 xP_1(x)dx = \frac{3}{2}\int_0^1 x^2dx = \frac{1}{2} \\ a_2 &= \frac{5}{2}\int_0^1 xP_2(x)dx = \frac{5}{2}\int_0^1 \frac{3x^3-x}{2}dx = \frac{5}{16}, \ a_3 = \frac{7}{2}\int_0^1 xP_3(x)dx = \frac{7}{2}\int_0^1 \frac{5x^4-3x^2}{2}dx = 0 \\ a_4 &= \frac{9}{2}\int_0^1 xP_4(x)dx = \frac{9}{2}\int_0^1 \frac{35x^5-30x^3+3}{8}dx = -\frac{3}{32} \end{aligned}$$

and so on. Using these values in (4.38), we get

$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \cdots$$

Example 4.2 If

$$f(x) = \begin{cases} 0; \text{ where } -1 < x < 0 \\ 1; \text{ where } 0 < x < 1 \end{cases}$$

Show that

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots$$

Solution: Given that

$$f(x) = \begin{cases} 0; \text{ where } -1 < x < 0 \\ 1; \text{ where } 0 < x < 1 \end{cases}$$

We know that

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{4.38}$$

$$\text{where } a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x) dx = \left(n + \frac{1}{2}\right) \int_{0}^{1} P_n(x) dx \tag{4.39}$$

Putting $n = 0, 1, 2, \cdots$ successively in (4.39), we get

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^1 P_0(x)dx = \frac{1}{2}\int_0^1 1dx = \frac{1}{2},\ a_1 = \frac{3}{2}\int_0^1 P_1(x)dx = \frac{3}{2}\int_0^1 xdx = \frac{3}{4} \\ a_2 &= \frac{5}{2}\int_0^1 P_2(x)dx = \frac{5}{2}\int_0^1 \frac{3x^2-1}{2}dx = 0,\ a_3 = \frac{7}{2}\int_0^1 P_3(x)dx = \frac{7}{2}\int_0^1 \frac{5x^3-3x}{2}dx = -\frac{7}{16} \end{aligned}$$

and so on. Using these values in (4.38), we get $f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots$.

4.7 Worked out Examples

Example 4.3 Show that

$$\int_{-1}^{1} x^2 P_{n+1}(x) P_{n-1}(x) dx = \frac{2n(n+1)}{(4n^2-1)(2n+3)}$$

Solution: From recurrence relation I, we get

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x) \tag{4.40}$$

Replacing n by $(n+2)$ in (4.40), we get

$$(2n+3)xP_{n+1}(x) = (n+2)P_{n+2}(x) + (n+1)P_n(x) \tag{4.41}$$

Multiplying corresponding side of (4.40) and (4.41), we get

$$(2n+3)(2n-1)x^2P_{n-1}(x)P_{n+1}(x) = n(n+1)P_n^2(x) + n(n+2)P_{n+2}(x)P_n(x)$$
$$+(n-1)(n+2)P_{n-2}(x)P_{n+2}(x) + (n-1)(n+1)P_{n-2}(x)P_n(x) \quad (4.42)$$

Integrating (4.42) with respect to x from -1 to 1, we get

$$(2n+3)(2n-1)\int_{-1}^{1} x^2P_{n-1}(x)P_{n+1}(x)dx = n(n+1)\int_{-1}^{1} P_n^2(x)dx + n(n+2)\int_{-1}^{1} P_{n+2}(x)P_n(x)dx$$
$$+(n-1)(n+2)\int_{-1}^{1} P_{n-2}(x)P_{n+2}(x)dx + (n-1)(n+1)\int_{-1}^{1} P_{n-2}(x)P_n(x)dx \quad (4.43)$$

Also we know that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0; \text{ if } m \neq n \\ \frac{2}{2n+1}; \text{ if } m = n \end{cases} \quad (4.44)$$

Using (4.44), (4.43) reduces to

$$(2n+3)(2n-1)\int_{-1}^{1} x^2P_{n-1}(x)P_{n+1}(x)dx = n(n+1)\frac{2}{2n+1}$$

$$\Rightarrow \int_{-1}^{1} x^2P_{n-1}(x)P_{n+1}(x)dx = \frac{2n(n+1)}{(2n+1)(2n+3)(2n-1)} = \frac{2n(n+1)}{(4n^2-1)(2n+3)}$$

Example 4.4 Prove that

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{(4n^2-1)}$$

Solution: From recurrence relation I, we get

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$
$$\Rightarrow \quad xP_n(x) = \frac{(n+1)}{(2n+1)}P_{n+1}(x) + \frac{n}{(2n+1)}P_{n-1}(x) \quad (4.45)$$

Multiplying corresponding side of (4.45) by $P_{n-1}(x)$, and then integrating both side with respect to $'x'$ from -1 to 1, we get

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{n+1}{2n+1}\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)dx + \frac{n}{2n+1}\int_{-1}^{1} \{P_{n-1}(x)\}^2dx \quad (4.46)$$

Also we know that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & ;\text{if } m \neq n \\ \frac{2}{2n+1} & ;\text{if } m = n \end{cases} \quad (4.47)$$

Using above result(4.46) reduces to

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2}{2(n-1)+1} \times \frac{n}{2n+1} \Rightarrow \int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$$

Example 4.5 Show that for any function $f(x)$, for which the $n-th$ derivative is continuous

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1}(1-x^2)^n f^{(n)}(x)dx$$

Solution: We know that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$. Now

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n f(x)dx$$

$$= \frac{1}{2^n n!}\left[\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n f(x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n f'(x)dx\right] \tag{4.48}$$

[On Intigration by parts]

Also $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = \frac{d^{n-1}}{dx^{n-1}}(x-1)^n(x+1)^n$

$= (x-1)^n(n-1)!(x+1) +^{(n-1)} C_1 n(x-1)^{n-1}(n-2)!(x+1)^2 + \cdots + (n-1)!(x-1)(x+1)^n$

[Since $(uv)_n = u_n v +^n C_1 u_{n-1}v_1 +^n C_2 u_{n-2}v_2 + \cdots + uv_n$, where $u_n = \frac{d^n u}{dx^n}$].

Now we can easily seen that $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ will be zero at $x = 1$ and $x = -1$. So first part of (4.48) must be zero, so from (4.48) reduces to

$$\int_{-1}^{1} f(x)P_n(x)dx = -\int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n f'(x)dx\Big] = (-1)^n\frac{1}{2^n n!}\int_{-1}^{1}(x^2-1)^n f^{(n)}(x)dx$$

$$= (-1)^n(-1)^n\frac{1}{2^n n!}\int_{-1}^{1}(1-x^2)^n f^{(n)}(x)dx = \frac{1}{2^n n!}\int_{-1}^{1}(1-x^2)^n f^{(n)}(x)dx.$$

Example 4.6 Show that, when $|h| < 1$ and $|x| \leq 1$

$$\int_{-1}^{1} P_n(x)(1-2xh+h^2)^{-\frac{1}{2}}dx = \frac{2h^n}{2n+1}$$

Solution: From generating function of Legendre polynomial, we get

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Multiplying both side by $P_n(x)$, we get

$$\Rightarrow \quad \frac{P_n(x)}{\sqrt{1-2xh+h^2}} = P_n(x)\Big[P_0(x) + hP_1(x) + \cdots + h^n P_n(x) + \cdots\Big]$$

Integrating both side w.r.t x between -1 to 1, we get

$$\Rightarrow \quad \int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-2xh+h^2}} dx = \int_{-1}^{1} P_n(x) P_0 dx + h \int_{-1}^{1} P_1 P_n dx + \cdots$$

$$+ \quad h^n \int_{-1}^{1} [P_n(x)]^2 dx + \int_{-1}^{1} P_n(x) P_{n+1} dx + \cdots$$

$$\Rightarrow \quad \int_{-1}^{1} P_n(x)(1-2xh+h^2)^{-\frac{1}{2}} dx = \frac{2h^n}{2n+1}\left[as \int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & ;\text{if } m \neq n \\ \frac{2}{2n+1} & ;\text{if } m = n \end{cases}\right]$$

Example 4.7 Prove that

$$(2n+1)(x^2-1)P_n'(x) = n(n+1)(P_{n+1}(x) - P_{n-1}(x))$$

Proof: From recurrence relations V and VI, we get

$$(1-x^2)P_n'(x) = n(P_{n-1}(x) - xP_n(x)) \tag{4.49}$$

$$(1-x^2)P_n'(x) = (n+1)(xP_n(x) - P_{n+1}(x)) \tag{4.50}$$

Multiplying (4.49) by $(n+1)$ and (4.50) by n and adding, we get

$$(n+1)(1-x^2)P_n'(x) + n(1-x^2)P_n'(x) = n(n+1)P_{n-1}(x) - n(n+1)P_{n+1}(x)$$

$$\Rightarrow \quad (2n+1)(1-x^2)P_n'(x) = n(n+1)(P_{n-1}(x) - P_{n+1}(x))$$

$$\Rightarrow \quad (2n+1)(x^2-1)P_n'(x) = n(n+1)(P_{n+1}(x) - P_{n-1}(x)).$$

Example 4.8 Prove that

$$\frac{1+h}{h\sqrt{(1-2xh+h^2)}} - \frac{1}{h} = \sum_{n=0}^{\infty} (P_n + P_{n+1})h^n$$

Solution: From generating function of Legendre polynomial, we get

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-\frac{1}{2}}$$

L.H.S of the required result

$$\text{Now} \frac{1+h}{h\sqrt{(1-2xh+h^2)}} - \frac{1}{h} = \frac{1}{h}(1-2xh+h^2)^{-\frac{1}{2}} + (1-2xh+h^2)^{-\frac{1}{2}} - \frac{1}{h}$$

$$= \frac{1}{h}\sum_{n=0}^{\infty} h^n P_n + \sum_{n=0}^{\infty} h^n P_n - \frac{1}{h} \tag{4.51}$$

$$\text{Now} \sum_{n=0}^{\infty} h^n Pn(x) = P_0 + hP_1 + h^2P_2 + \cdots + h^nP_n + h^{n+1}P_{n+1} + \cdots$$

$$= 1 + h(P_1 + hP_2 + \cdots + h^nP_{n+1} + \cdots) = 1 + h\sum_{n=0}^{\infty} h^nP_{n+1} \tag{4.52}$$

Using (4.51) and (4.52), we get

$$\begin{aligned}\sum_{n=0}^{\infty} h^nP_n &= \frac{1}{h}\Big[1 + h\sum h^nP_{n+1}\Big] + \sum_{n=0}^{\infty} h^nP_n - \frac{1}{h}\\ &= \sum_{n=0}^{\infty} h^nP_{n+1} + \sum_{n=0}^{\infty} h^nP_n = \sum_{n=0}^{\infty}\Big(P_n + P_{n+1}\Big)h^n.\end{aligned}$$

Example 4.9 Prove that

$$P'_{n+1}(x) + P'_n(x) = P_0(x) + 3P_1(x) + 5P_2(x) + 7P_3(x) + \cdots + (2n+1)P_n(x)$$

Solution: From recurrence relation III, we get

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \tag{4.53}$$

Replacing n by $1, 2, \cdots, (n-1), n$ successively in (4.53) we get

$$\begin{aligned}3P_1(x) &= P'_2(x) - P'_0(x)\\ 5P_2(x) &= P'_3(x) - P'_1(x)\\ 7P_3(x) &= P'_4(x) - P'_2(x)\\ \cdots & \qquad \cdots \quad \cdots\\ (2n-1)P_{n-1}(x) &= P'_n(x) - P'_{n-2}(x)\\ (2n+1)P_n(x) &= P'_{n+1}(x) - P'_{n-1}(x)\end{aligned}$$

Adding these and noting that in the sum of right hand sides all the terms cancel except the first two of the second column and the last two of the first column, we get

$$3P_1(x) + 5P_2(x) + 7P_3(x) + \cdots + (2n+1)P_n(x) = -P'_0(x) - P'_1(x) + P'_n(x) + P'_{n+1}(x)$$

Now since $P_0(x) = 1$ and $P_1(x) = x$, so $P'_0 = 0$ and $P'_1(x) = 1$, we get

$$P_0(x) + 3P_1(x) + 5P_2(x) + 7P_3(x) + \cdots + (2n+1)P_n(x) = P'_n(x) + P'_{n+1}(x)$$

Example 4.10 Prove that

$$1 + \tfrac{1}{2}P_1(\cos\theta) + \tfrac{1}{3}P_2(\cos\theta) + \cdots = \log\left[\frac{(1+\sin\frac{\theta}{2})}{\sin\frac{\theta}{2}}\right]$$

Solution: From generating function, we get

$$(1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^nP_n(x), |x| \le 1, |h| < 1$$

Integrating with respect to 'h' from 0 to 1, we get

$$\int_0^1 (1 - 2xh + h^2)^{-\frac{1}{2}}dh = \int_0^1 \sum_{n=0}^{\infty} h^nP_n(x)dh, \tag{4.54}$$

Replacing x by $\cos\theta$ on both sides, (4.54) reduces to

$$\int_0^1 (1-2\cos\theta h+h^2)^{-\frac{1}{2}}dh=\sum_{n=0}^{\infty}P_n(\cos\theta)\int_0^1 h^n dh$$

$$\Rightarrow\quad \int_0^1 \frac{1}{\sqrt{[(h-\cos\theta)^2+\sin^2\theta]}}dh=\sum_{n=0}^{\infty}P_n(\cos\theta)\left[\frac{h^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right]=\left[\log\{(h-\cos\theta)+\sqrt{[(h-\cos\theta)^2+\sin^2\theta]}\}\right]_0^1$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right]=\log\left\{(1-\cos\theta)+\sqrt{[(1-\cos\theta)^2+\sin^2\theta]}\right\}-\log(1-\cos\theta)$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right]=\log\left\{(1-\cos\theta)+\sqrt{[2(1-\cos\theta)]}\right\}-\log(1-\cos\theta)$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right]=\log\frac{\{(1-\cos\theta)+\sqrt{[2(1-\cos\theta)]}\}}{(1-\cos\theta)}$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right]=\log\frac{\{\sqrt{(1-\cos\theta)}\sqrt{(1-\cos\theta)}+\sqrt{[2(1-\cos\theta)]}\}}{\sqrt{(1-\cos\theta)}\sqrt{(1-\cos\theta)}}$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right]=\log\frac{\sqrt{(1-\cos\theta)}+\sqrt{2}}{\sqrt{(1-\cos\theta)}}$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right]=\log\frac{\sqrt{2\sin^2\frac{\theta}{2}}+\sqrt{2}}{\sqrt{2\sin^2\frac{\theta}{2}}}$$

$$\Rightarrow\quad \sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right]=\log\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\Rightarrow\quad P_0(\cos\theta)+\frac{1}{2}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+\cdots=\log\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\Rightarrow\quad 1+\frac{1}{2}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+\cdots=\log\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

Example 4.11 Find the value of $\int_{-1}^{1} P_0(x)dx$ where $P_n(x)$ is Legendre's polynomial of degree n.

Solution: We know that $P_0(x)=1$. So $\int_{-1}^{1} P_0(x)dx=\int_{-1}^{1} dx=2$.

Example 4.12 Express x^3+x^2 in terms of Legendre polynomials $P_0(x), P_1(x), P_2(x), P_3(x)$.

Solution: We know that

$$P_0(x) = \frac{1}{0!}x^0 = 1 \tag{4.55}$$
$$P_1(x) = x \tag{4.56}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3} \tag{4.57}$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x \tag{4.58}$$

Now using (4.57) and (4.58), $x^3 + x^2$ reduces to

$$\begin{aligned} x^3 + x^2 &= \frac{2}{5}P_3(x) + \frac{3}{5} + \frac{2}{3}P_2(x) + \frac{1}{3} = \frac{2}{5}P_3(x) + \frac{2}{3}P_2(x) + \frac{3}{5}x + \frac{1}{3} \\ &= \frac{2}{5}P_3(x) + \frac{2}{3}P_2(x) + \frac{3}{5}P_1(x) + \frac{1}{3}P_0(x), \text{ Since } P_1(x) = x \text{ and } P_0(x) = 1. \end{aligned}$$

Example 4.13 Express $x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

Solution: We know that

$$P_0(x) = \frac{1}{0!}x^0 = 1 \tag{4.59}$$
$$P_1(x) = x \tag{4.60}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}. \tag{4.61}$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x \tag{4.62}$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \Rightarrow x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35} \tag{4.63}$$

Now using (4.63) and (4.62) $x^4 + 2x^3 + 2x^2 - x - 3$ reduces to

$$\begin{aligned} x^4 + 2x^3 + 2x^2 - x - 3 &= \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35} + 2\left\{\frac{2}{5}P_3(x) + \frac{3}{5}x\right\} + 2x^2 - x - 3 \\ &= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{20}{7}x^2 + \frac{1}{5}x - \frac{108}{35} \\ &= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) - \frac{20}{7}\left\{\frac{2}{3}P_2(x) + \frac{1}{3}\right\} + \frac{1}{5}P_1(x) - \frac{108}{35}, \text{ [By (4.61)]} \\ &= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{224}{105}, \ [\because P_1(x) = x] \\ &= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{224}{105}P_0(x), \ [\because P_0(x) = 1] \end{aligned}$$

Example 4.14 Show that all the roots of $P_n(x) = 0$ are real and lie between -1 and 1.

Solution: We get Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left\{(x^2 - 1)^n\right\}$$

Now, $(x^2-1)^n = (x-1)^n(x+1)^n$. Since $(x^2-1)^n$ vanishes n times at $x=1$ and n times at $x=-1$, therefore, using the theory of equations $\frac{d^n}{dx^n}(x^2-1)^n = 0$ will get n roots all lying between -1 and 1. Therefore $P_n(x)=0$ has n real roots all lying between -1 and 1.

Example 4.15 Express $x^4 - 3x^2 + x$ in terms of Legendre polynomials $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)$.

Solution: We know that

$$P_0(x) = \frac{1}{0!}x^0 = 1 \tag{4.64}$$

$$P_1(x) = x \tag{4.65}$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}. \tag{4.66}$$

$$P_3(x) = \frac{1}{2}(5x^3-3x) \Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x \tag{4.67}$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3) \Rightarrow x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35} \tag{4.68}$$

Now using (4.68) $x^4 - 3x^2 + x$ reduces to

$$\begin{aligned}
x^4 - 3x^2 + x &= \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35} - 3x^2 + x \\
&= \frac{8}{35}P_4(x) - \frac{15}{7}x^2 - \frac{3}{35} + x \\
&= \frac{8}{35}P_4(x) - \frac{15}{7}\left\{\frac{2}{3}P_2(x) + \frac{1}{3}\right\} - \frac{3}{35} + x, \quad \text{[Using (4.66)]} \\
&= \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) - \frac{5}{7} - \frac{3}{35} + P_1(x), \quad \text{[Since } P_1(x) = x\text{]} \\
&= \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x), \quad \text{[Since } P_0(x) = 1\text{]}
\end{aligned}$$

Example 4.16 Express $4x^3 + 6x^2 + 7x + 2$ in terms of Legendre polynomials.

Solution: We know that

$$P_0(x) = \frac{1}{0!}x^0 = 1 \tag{4.69}$$

$$P_1(x) = x \tag{4.70}$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3} \tag{4.71}$$

$$P_3(x) = \frac{1}{2}(5x^3-3x) \Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x \tag{4.72}$$

Now using (4.72) $4x^3 + 6x^2 + 7x + 2$ reduces to

$$\begin{aligned}
4x^3 + 6x^2 + 7x + 2 &= 4\{\frac{2}{5}P_3(x) + \frac{3}{5}x\} + 6x^2 + 7x + 2 = \frac{8}{5}P_3(x) + 6x^2 + \frac{12}{5}x + 7x + 2 \\
&= \frac{8}{5}P_3(x) + 6\{\frac{2}{3}P_2(x) + \frac{1}{3}\} + \frac{12}{5}x + 7x + 2, \quad \text{[Using (4.71)]} \\
&= \frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4, \quad \text{[Since } P_1(x) = x\text{]} \\
&= \frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x), \quad \text{[Since } P_0(x) = 1\text{]}
\end{aligned}$$

Example 4.17 Evaluate $\int\limits_{-1}^{1} P_3^2(x)dx$.

Solution: Since $\int\limits_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$, so $\int\limits_{-1}^{1} P_3^2(x)dx = \frac{2}{2\times 3+1} = \frac{2}{7}$.

Example 4.18 Evaluate $\int\limits_{-1}^{1} P_0(x)dx$.

Solution: Since $P_0(x) = 1$ so, $\int\limits_{-1}^{1} P_0(x)dx = \int\limits_{-1}^{1} dx = 2$.

4.8 Multiple Choice Questions(MCQ)

1. Let $P_n(x)$ be the Legedre polynomial of degree n and $I = \int_{-1}^{1} x^k P_n(x)dx$, where k is the non-negative integer. Consider the following statements P and Q : **GATE(MA)-2016**
   (P) : $I = 0$ if $k < n$.
   (Q) : $I = 0$ if $n - k$ is an odd integer.
   which of the following statements hold TRUE?
   (A) both P and Q (B) only P (C) only Q (D) Neither P nor Q.
   Ans. (A).
   Hint. We have $x^k = \sum\limits_{m=0}^{k} C_m P_m(x)$ where C_m are real constants. Also $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$ if $m \neq n$. Hence the result.
2. Let the Legedre equation

   $$(1 - x^2)y^{''} - 2xy^{'} + n(n+1)y = 0$$

   have n-th degree polynomial solution $y_n(x)$ such that $y_n(1) = 3$. If $\int_{-1}^{1}(y_n^2(x) + y_{n-1}^2(x))dx = \frac{144}{15}$, then n is **GATE(MA)-12**
   A) 1 B) 2 C) 3 D) 4.
   Ans. B)
3. Let $P_n(x)$ be the Legendre polynomial of degree n such that $P_n(1) = 1$, $n = 1, 2, \cdots$ if $\int_{-1}^{1}\left(\sum\limits_{j=1}^{n} \sqrt{j(2j+1)}P_j(x)\right)^2 dx = 20$, then $n =$ **GATE(MA)-09**
   A) 2 B) 3 C) 4 D) 5.
   Ans. C)
   Hint. $\int_{-1}^{1}(P_n(x))^2 dx = \frac{2}{2n+1}$
4. Let $P_n(x)$ be the Legendre polynomial of degree n and let

   $$P_{m+1}(0) = -\frac{m}{m+1}P_{m-1}(0), \quad m = 1, 2, \ldots$$

   If $P_n(0) = -\frac{5}{16}$, then $\int_{-1}^{1} P_n^2(x)dx =$ **GATE(MA)-07**
   A) $\frac{2}{13}$ B) $\frac{2}{9}$ C) $\frac{5}{16}$ D) $\frac{2}{5}$.
   Ans. A)

Hint. $P_1(0) = 0,\ P_2(0) = -\frac{1}{2}P_0(0) = -\frac{1}{2},\ P_3(0) = -\frac{2}{3}P_1(0) = 0, \cdots, P_6(0) = -\frac{5}{16}$
$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1} = \frac{2}{13}$

5. The weight function of Legendre polynomial is
(a) $W(x) = 1$ (b) $W(x) = x$ (c) $W(x) = 1 - x$ (d) none of these.
Ans. (a) $W(x) = 1$

6. Let $P_n(x)$ denote the Legendre polynomial of degree n. If

$$f(x) = \begin{cases} x, & -1 \le x \le 0 \\ 0, & 0 \le x \le 1 \end{cases}$$

and $f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \ldots$ then **GATE(MA)-05**

A) $a_0 = -\frac{1}{4},\ a_1 = -\frac{1}{2}$ B) $a_0 = -\frac{1}{4},\ a_1 = \frac{1}{2}$
C) $a_0 = \frac{1}{2},\ a_1 = -\frac{1}{4}$ D) $a_0 = -\frac{1}{2},\ a_1 = -\frac{1}{4}$.
Ans. B)
Hint. $f(x) = \sum_{r=0}^{\infty} a_rP_r(x),\ a_r = (r + \frac{1}{2}) \int_{-1}^{1} f(x)P_r(x)dx$

7. Let $P_n(x)$ be the Legendre polynomial of degree $n \le 0$. If $1 + x^{10} = \sum_{n=0}^{10} C_nP_n(x)$, then C_5 is **GATE(MA)-04**

A) 0 B) $\frac{2}{11}$ C) 1 D) $\frac{11}{2}$.
Ans. A)
Hint. As equating the co-efficient of x^5.

8. Let $y = \phi(x)$ and $y = \psi(x)$ be solutions of

$$y'' - 2xy' + (\sin x^2)y = 0$$

such that $\phi(0) = 1\ \phi'(0) = 1,\ \psi(0) = 1,\ \psi'(0) = 2$. Then the value of $W(\phi, \psi)$ at $x = 1$ is **GATE(MA)-04**

A) 0 B) 1 C) e D) e^2.
Ans. C)

9. Let$y$ be the polynomial solution of the differential equation

$$(1 - x^2)y'' - 2xy' + 6y = 0$$

If $y(1) = 2$, then the value of the integral $\int_{-1}^{1} y^2(x)dx$ is **GATE(MA)-11**
A) $\frac{1}{5}$ B) $\frac{2}{5}$ C) $\frac{4}{5}$ D) $\frac{8}{5}$.
Ans. D)
Hint. $I = y(1)^2\frac{2}{2n+1}$

10. The interval of x of Legendre polynomial is
(a) $[-1, 1]$ (b) $(-1, 1)$ (c) $[0, 1]$ (d) $[-1, 1)$
Ans. (a) $[-1, 1]$.

11. The Legendre polynomial $P_n(x)$ is
(a) even if n is even (b) odd if n is even (c) even if n is odd (d) none of these.
Ans. (a) even if n is even.

4.9 Review Exercise

1 Show that infinity is a regular singular point for the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

where α is constant.

1. The Laguerre polynomial

$$L_n(x) = e^x \frac{d^(x^n e^{-x})}{dx^n}$$

is a solution of the Laguerre equation.

2. The Laguerre polynomials L_n are orthogonal with respect to the weight function $w(x) = e^{-x}$, in the sense that

$$\int_0^{+\infty} e^{-x} L_m(x) L_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

3. Prove that $(i) P'_n(-1) = (-1)^n \frac{1}{2} n(n+1)$
$(ii) \int_{-1}^{1} (P'_n)^2 dx = n(n+1)$.
4. Prove that $\int_{-1}^{1} x P_n(x) P'_n(x) dx = \frac{2n}{2n+1}$.
5. prove that $P'_n(x) - P'_{n-1}(x) = (2n-1)P_{n-1}$.
6. Express $2 - 3x + 4x^2$ in terms of Legendre polynomial.
7. Show that $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_2(x)$.
8. Prove that $\int_0^{\pi} P_n(cos\theta) cosn\theta d\theta = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots(2n)}\pi$.
9. Prove that $(2n+1)(x^2-1)P'_n(x) = n(n+1)(P_{n+1}(x) - P_{n-1}(x))$.
10. Prove that $P'_{n+1} + P'_n = P_0 + 3P_1 + \cdots + (2n+1)P_n$.
11. Using Rodrigue's formula, prove that $P'_{n+1} - P'_{n-1} = (2n+1)P_n$.
12. Let P be any polynomial of degree n and let

$$P = c_0 P_0 + c_1 P_1 + \cdots + c_n P_n,$$

where $c_0, c_1, \cdots, c_n$ are constants. Show that $c_k = \frac{2k+1}{2} \int_{-1}^{1} P(x) P_k(x) dx$, $(k = 0, 1, 2, \cdots, n)$.

13. Prove that $P_1(x) = \frac{1}{\pi} \int_0^{\pi} \{x + \sqrt{x^2-1} cos\theta\}$.

14. Express $P(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of legendre polynomials.
[**Ans**. $8/35P_4(x) + 4/5P_3(x) + 40/21P_2(x) + 1/5P_1(x) - 224/105P_0(x)$].
15. Prove that all the roots of $p_n(x)$ are distinct.
16. Show that all the roots of $P_n(x)$ are real and lie between -1 and 1.
17. Using Rodrigue's formula find the value of $P_0(x)$, $P_1(x)$, $P_2(x)$ and $P_3(x)$.
18. Show that $\int_{-1}^{1} x^n P_n(x)dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}$.
19. Show that $\int_{-1}^{1} x^4 P_6(x)dx = 0$.
20. Prove that $\int_{0}^{\pi} P_n(\cos\theta)\cos n\theta d\theta = B(n+\frac{1}{2}, \frac{1}{2})$.
21. When n is a positive integer prove that $P_n(x) = \frac{1}{\pi}\int_{0}^{\pi}[x \pm \sqrt{x^2-1}\cos\phi]^n d\phi$.
22. Using Rodrigues formula to prove that $P'_{n+1} - P'_{n-1} = (2n+1)P_n$.

Chapter 5

Bessel Functions

5.1 Introduction

Another special type of differential equation is discussed in this chapter which is called Bessel equation. These are named after the German mathematician and astronomer Friedrich Bessel, who first used them to analyze planetary orbits which will discussed later. Bessel functions occur in many other physical problems, usually in a cylindrical geometry, wave propagation and static potentials. Some examples of these are presented at the end of this chapter.

5.2 Bessel Equation

The differential equation of the form

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{5.1}$$

is called Bessel equation of order n, n being a non negative constant. Comparing the equation (5.1) with the equation (3.1), we get $p_1(x) = \frac{1}{x}$ and $p_0(x) = \frac{x^2-n^2}{x^2}$. Obviously, $x = 0$ is a singular point. Note that $\lim_{x\to 0} xp_1(x) = 1$ and $\lim_{x\to 0} x^2p_0(x) = -n^2$. So, both $xp_1(x)$ and $x^2p_0(x)$ are analytic at $x = 0$ and can be expanded as power series that are convergent for $|x| < \infty$. Hence, $x = 0$ is a regular singular point. To investigate the point at $x = \infty$, we transfer the independent variable x by $x = \frac{1}{t}$, then $\frac{dy}{dx} = -t^2\frac{dy}{dt}$, $\frac{d^2y}{dt^2} = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$ and subsequently, the equation (5.1) becomes

$$t^4\frac{d^2y}{dt^2} + t^3\frac{dy}{dt} + (1 - n^2t^2)y = 0 \tag{5.2}$$

This shows that the equation (5.2) has a singular point at $t = 0$ which is not regular singular point as $\lim_{t\to 0} t^2p_0(t) = \lim_{t\to 0} \frac{t^2(1-n^2t^2)}{t^4}$ does not exist and hence $x = \infty$ is not a regular singular point of the equation (5.1).

To obtain a series solution of the above differential equation (5.1) in the neighborhood of $x = 0$ by Frobenius method, let us put

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}, a_0 \neq 0,\ 0 < x < \infty \tag{5.3}$$

Differentiating twice (5.3) in a succession, we get

$$y' = \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} \text{ and } y'' = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}$$

Putting these values y, y' and y'' in (5.1) we get

$$x^2\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} + x\sum_{n=0}^{\infty}(m+r)a_m x^{m+r-1} + (x^2-n^2)\sum_{m=0}^{\infty}a_m x^{m+r} = 0$$
$$\Rightarrow \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r} + \sum_{m=0}^{\infty}a_m x^{m+r+2} - n^2\sum_{m=0}^{\infty}a_m x^{m+r} = 0$$
$$\Rightarrow \sum_{m=0}^{\infty}\{(m+r)(m+r-1)+(m+r)-n^2\}a_m x^{m+r} + \sum_{m=0}^{\infty}a_m x^{m+r+2} = 0$$

Equating to zero the coefficient of smallest power of x, namely x^r, the indicial equation is

$$a_0\{r(r-1)+r-n^2\} = 0, \quad \text{i.e. } r^2-n^2=0 \text{ as } a_0 \neq 0$$

So the roots of the indicial equation are $r = -n$ and n. Next equating to zero the coefficient of x^{k+r} from above equation, we obtain the recurrence relation as

$$(k+r+n)(k+r-n)a_k + a_{k-2} = 0 \Rightarrow a_k = -\frac{a_{k-2}}{(k+r+n)(k+r-n)}, \; k = 2,3,4,\cdots \tag{5.4}$$

Next equating to zero the coefficient of x^{r+1} and get

$$a_1(r+n+1)(r+1-n) = 0 \Rightarrow a_1 = 0 \text{ (for } r=-n \text{ and } r=n). \tag{5.5}$$

Using $a_1 = 0$ and (5.4), we get $a_1 = a_3 = a_5 = a_7 = \cdots = 0$. Putting $n = 2,4,6,\cdots$ in (5.4) we get

$$a_2 = -\frac{a_0}{(r+n+2)(r+2-n)},$$
$$a_4 = -\frac{a_2}{(r+n+4)(r+4-n)} = \frac{a_0}{(r+n+4)(r+4-n)(r+n+2)(r+2-n)}$$

and so on. Putting these values in (5.3), we get

$$y(x) = a_0\left[x^r - \frac{x^{r+2}}{(r+n+2)(r+2-n)} + \frac{x^{r+4}}{(r+n+4)(r+4-n)(r+n+2)(r+2-n)} + \cdots\right] \tag{5.6}$$

Putting $r = n$ and replacing a_0 by a in (5.6), we get

$$(y)_{r=n} = ax^n\left[1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4.8(1+n)(2+n)} - \cdots\right]$$

If $a = \frac{1}{2^n\Gamma(n+1)}$, then the solution is

$$J_n(x) = \frac{1}{2^n\Gamma(n+1)}x^n\left[1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4.8(1+n)(2+n)} - \cdots\right] = \sum_{m=0}^{\infty}(-1)^m\frac{1}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}$$

which is called the Bessel's function of first kind of order n. Putting $r = -n$ and replacing a_0 by b in (5.6), we get

$$(y)_{r=-n} = bx^{-n}\left[1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4.8(1-n)(2-n)} - \cdots\right]$$

If $b = \frac{1}{2^n\Gamma(n+1)}$, then the solution is

$$J_{-n}(x) = \frac{1}{2^n\Gamma(n+1)}x^{-n}\left[1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4.8(1-n)(2-n)} - \cdots\right] = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{2m-n}$$

which is called the Bessel's function of second kind of order $-n$. Thus the general solution of Bessel equation (5.1) when n is not an integer is $y = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constant.

5.3 Bessel's function of first kind of order n

The bessel's function of first kind of order n is denoted by $J_n(x)$ and is defined as

$$\begin{aligned} J_n(x) &= \frac{1}{2^n\Gamma(n+1)}x^n\left[1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4.8(1+n)(2+n)} - \cdots\right] \\ &= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n} \end{aligned} \tag{5.7}$$

where n is a nonnegative constant. When n is a integer, $\Gamma(n+r+1) = (n+r)!$ and so (5.7) may be written as

$$J_n(x) = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!(n+m)!}\left(\frac{x}{2}\right)^{2m+n}$$

5.4 Properties of Bessel's function

Theorem 5.1 Show that for any integer n, $J_{-n}(x) = (-1)^n J_n(x)$ and $J_n(x)$ is Bessel function of first kind.

Proof: Case-I: When n is a positive integer
From the Bessel's function of first kind, we have

$$J_n(x) = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}(\frac{x}{2})^{2m+n}$$

Now replacing n by $-n$ in the above expression, we have

$$J_{-n}(x) = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(-n+m+1)}\left(\frac{x}{2}\right)^{2m-n} \tag{5.8}$$

since $n > 0$, so $\Gamma(-n+m+1)$ is infinite and so $\frac{1}{\Gamma(-n+m+1)}$ is zero for $m = 0, 1, 2, \cdots (n-1)$. Keeping this in mind we see that the sum over the m in (5.8) must be taken from n to ∞. Thus we obtain

$$J_{-n}(x) = \sum_{m=n}^{\infty}(-1)^m \frac{1}{m!\Gamma(-n+m+1)}\left(\frac{x}{2}\right)^{2m-n}$$

Taking $r = m - n$, we get

$$\begin{aligned} J_{-n}(x) &= \sum_{r=0}^{\infty}(-1)^{r+n}\frac{1}{(r+n)!\Gamma(r+1)}\left(\frac{x}{2}\right)^{2(r+n)-n} = \sum_{r=0}^{\infty}(-1)^{r}(-1)^{n}\frac{1}{r!\Gamma(r+n+1)}\left(\frac{x}{2}\right)^{2r+n} \\ &= (-1)^{n}\sum_{r=0}^{\infty}(-1)^{r}\frac{1}{r!\Gamma(r+n+1)}\left(\frac{x}{2}\right)^{2r+n} = (-1)^{n}J_n(x) \end{aligned}$$

Case-II: When n is a negative integer
Let p be a positive integer such that $n = -p$. Since $p > 0$, from case-I, we have $J_{-p}(x) = (-1)^p J_p(x) \Rightarrow J_p(x) = (-1)^{-p}J_{-p}(x)$. But $p = -n$ hence above result becomes $J_{-n}(x) = (-1)^n J_n(x)$. Hence for any integer n, $J_{-n}(x) = (-1)^n J_n(x)$.

Theorem 5.2 Show that for any integer n, $J_n(-x) = (-1)^n J_n(x)$.

Proof:Case-I: When n is a positive integer
From the Bessel's function of first kind, we have $J_n(x) = \sum\limits_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}$.
Now replacing x by $-x$ in the above expression, we have

$$\begin{aligned} J_n(-x) &= \sum_{m=0}^{\infty}(-1)^{m}\frac{1}{m!\Gamma(n+m+1)}\left(\frac{-x}{2}\right)^{2m+n} = \sum_{m=0}^{\infty}(-1)^{m}(-1)^{2m+n}\frac{1}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n} \\ &= (-1)^{n}\sum_{m=0}^{\infty}(-1)^{m}\frac{1}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n} = (-1)^{n}J_n(x) \end{aligned}$$

Case-II: When n is a negative integer
Let p be a positive integer such that $n = -p$. Since $p > 0$, from case-I of theorem 5.1, we have

$$J_n(x) = J_{-p}(x) = (-1)^p J_p(x) \;\; [\text{Since } J_{-n}(x) = (-1)^n J_n(x)]$$

Now replacing x by $-x$, we have

$$\begin{aligned} J_n(-x) &= (-1)^p J_p(-x) = (-1)^{2p}J_p(x), \;\; [\text{Since } J_n(-x) = (-1)^n J_n(x) \text{ for any positive integer } n] \\ &= J_p(x) = (-1)^{-p}J_{-p}(x) = (-1)^n J_n(x), \;\; [\text{Since } J_{-n}(x) = (-1)^n J_n(x) \Rightarrow J_n(x) = (-1)^{-n}J_{-n}(x)\,] \end{aligned}$$

Hence for any integer, n, $J_n(-x) = (-1)^n J_n(x)$.

Theorem 5.3 (**Recurrence Relation I**) Prove that

$$\frac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x) \text{ or } xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$$

Solution: From the Bessel's function of first kind, we have

$$J_n(x) = \sum_{m=0}^{\infty}(-1)^{m}\frac{1}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n} \Rightarrow x^n J_n(x) = \sum_{m=0}^{\infty}(-1)^{m}\frac{1}{m!\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m+n}x^{2m+2n}$$

Differentiating with respect to x both side we get

$$\begin{aligned}
\frac{d}{dx}\{x^n J_n(x)\} &= \frac{d}{dx}\left\{\sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m+n} x^{2m+2n}\right\} \\
&= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m+n} \frac{d}{dx}x^{2m+2n} \\
&= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}(2m+2n)\left(\frac{1}{2}\right)^{2m+n} x^{2m+2n-1} \\
&= x^n \sum_{m=0}^{\infty}(-1)^m \frac{1}{\Gamma(m+1)\Gamma(n+m+1)} 2(m+n)\left(\frac{1}{2}\right)^{2m+n} x^{n-1+2m} \\
&= x^n \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m)}\left(\frac{1}{2}\right)^{n-1+2m} x^{n-1+2m} \\
&= x^n \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma\{(n-1)+m+1\}}\left(\frac{x}{2}\right)^{n-1+2m} = x^n J_{n-1}(x).
\end{aligned}$$

Hence $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x) \Rightarrow x^n J_n'(x) + nx^{n-1}J_n(x) = x^n J_{n-1}(x) \Rightarrow xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$.

Theorem 5.4 (**Recurrence Relation II**) Prove that

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x) \qquad \text{or} \quad xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Solution: From the Bessel's function of first kind, we have

$$\begin{aligned}
J_n(x) &= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n} \\
\Rightarrow \quad x^{-n}J_n(x) &= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m+n} x^{2m}
\end{aligned}$$

Differentiating with respect to x both side we get,

$$\begin{aligned}
\frac{d}{dx}\{x^{-n}J_n(x)\} &= \frac{d}{dx}\{\sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m+n} x^{2m}\} \\
&= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m+n}\frac{d}{dx}x^{2m} = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m+n} 2mx^{2m-1} \\
&= x^{-n}\sum_{m=1}^{\infty}(-1)^m \frac{1}{\Gamma(m)\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m+n-1} x^{n+2m-1} \quad [\text{Since } \Gamma(m) = \infty \text{ when } m = 0]
\end{aligned}$$

$$= \quad x^{-n}\sum_{m=1}^{\infty}(-1)^m \frac{x^{\{(n+1)+2(m-1)\}}}{\Gamma\{(m-1)+1\}\Gamma\{(n+1)+(m-1)+1)\}}\left(\frac{1}{2}\right)^{n-1+2m}$$

$$= \quad x^{-n}\sum_{r=0}^{\infty}(-1)^{r+1} \frac{1}{\Gamma(r+1)\Gamma\{(n+1)+r+1\}}\left(\frac{x}{2}\right)^{\{n-1+2(r+1)\}}, \text{ [Replacing } r = m-1]$$

$$= \quad -x^{-n}\sum_{r=0}^{\infty}(-1)^{r} \frac{1}{r!\Gamma\{(n+1)+r+1\}}\left(\frac{x}{2}\right)^{(n+1+2r)} = -x^{-n}J_{n+1}(x).$$

Hence $\dfrac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x)$

$\Rightarrow x^{-n}J_n'(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x) \Rightarrow xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x).$

Theorem 5.5 (**Recurrence Relation III:**) Prove that

$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$, where $J_n(x)$ is the Bessel's function of first kind.

Proof: From recurrence relations I and II, we have

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \tag{5.9}$$

$$\text{and } xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \tag{5.10}$$

Adding (5.9) and (5.10), we get

$$2xJ_n'(x) = xJ_{n-1}(x) - xJ_{n+1}(x) \Rightarrow 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Theorem 5.6 (**Recurrence Relation IV:**)Prove that

$2nJ_n(x) = x\Big[J_{n-1}(x) + J_{n+1}(x)\Big]$, where $J_n(x)$ is the Bessel's function of first kind.

Proof: From recurrence relations I and II, we have

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \tag{5.11}$$

$$\text{and } xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \tag{5.12}$$

Subtracting (5.12) from (5.11), we get

$$2nJ_n(x) - xJ_{n+1}(x) - xJ_{n-1}(x) = 0 \Rightarrow 2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)].$$

5.5 Generating Function for the Bessel's function $J_n(x)$

Example 5.1 Show that when n is a positive integer, $J_n(x)$ is the coefficient of z^n in the expansion of $\exp\left\{\frac{x}{2}(z-\frac{1}{z})\right\}$, i.e $\exp\{\frac{x}{2}(z-\frac{1}{z})\} = \sum\limits_{n=-\infty}^{\infty} z^n J_n(x)$

Solution: We have $\exp\left\{\frac{x}{2}(z-\frac{1}{z})\right\} = e^{\frac{xz}{2}-\frac{x}{2z}} = e^{\frac{xz}{2}}e^{-\frac{x}{2z}}$

$$\begin{aligned}
\text{Now, } \exp\left\{\frac{x}{2}(z-\frac{1}{z})\right\} &= e^{\frac{xz}{2}-\frac{x}{2z}} = e^{\frac{xz}{2}}e^{-\frac{x}{2z}} \\
&= \left[1+\left(\frac{x}{2}\right)z+\left(\frac{x}{2}\right)^2\frac{z^2}{2!}+\cdots+\left(\frac{x}{2}\right)^n\frac{z^n}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{z^{(n+1)}}{(n+1)!}+\cdots\right] \\
&\times \left[1-\left(\frac{x}{2}\right)z^{-1}+\left(\frac{x}{2}\right)^2\frac{z^{-2}}{2!}+\cdots+\left(\frac{x}{2}\right)^n\frac{(-1)^n z^{-n}}{n!}\right. \\
&+ \left.\left(\frac{x}{2}\right)^{n+1}\frac{(-1)^{n+1}z^{-(n+1)}}{(n+1)!}+\cdots\right] \qquad (5.13)
\end{aligned}$$

Now the coefficient of z^n in the product of (5.13) is obtained by multiplying coefficients of $z^n, z^{n+1}, z^{n+2}, \cdots$ in the first brackets with coefficients of $z^0, z^{-1}, z^{-2}, \cdots$ in the second bracket respectively and in thus

$$\begin{aligned}
&= \left(\frac{x}{2}\right)^n\frac{1}{n!}-\left(\frac{x}{2}\right)^{n+2}\frac{1}{(n+1)!}+\left(\frac{x}{2}\right)^{n+4}-\cdots \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{n+2m} \quad \left[\text{Since } (n+m)! = \Gamma(n+m+1),\ n+m \text{ being positive integer}\right] \\
&= J_n(x)
\end{aligned}$$

Example 5.2 Prove that the following recursion relation for Bessel's function using generating function

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

Solution: From generating function of Bessel's function we have

$$exp\{\tfrac{1}{2}x(z-\tfrac{1}{z})\} = \sum_{n=-\infty}^{\infty} z^n J_n(x).$$

Differentiating with respect to $'z'$ both side, we get

$$\begin{aligned}
&exp\left\{\frac{1}{2}x(z-\frac{1}{z})\right\}\frac{x}{2}\left(1+\frac{1}{z^2}\right) = \sum_{n=-\infty}^{\infty} nz^{n-1}J_n(x) \\
\Rightarrow\ & \frac{x}{2}\left(1+\frac{1}{z^2}\right)\sum_{n=-\infty}^{\infty} z^n J_n(x) = \sum_{n=-\infty}^{\infty} nz^{n-1}J_n(x) \\
\Rightarrow\ & \frac{x}{2}\sum_{n=-\infty}^{\infty} z^n J_n(x) + \frac{x}{2}\sum_{n=-\infty}^{\infty} z^{n-2}J_n(x) = \sum_{n=-\infty}^{\infty} nz^{n-1}J_n(x)
\end{aligned}$$

Equating the coefficient of z^{n-1} both side we get

$$\frac{x}{2}J_{n-1}(x) + \frac{x}{2}J_{n+1}(x) = nJ_n(x) \ \Rightarrow\ 2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)].$$

Example 5.3 Prove that the following recursion relation for Bessel,s function using generating function

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Solution: From the generating function of Bessel's function we know that

$$exp\{\tfrac{1}{2}x(z - \tfrac{1}{z})\} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

Differentiating with respect to $'x'$ both side, we get

$$exp\left\{\frac{1}{2}x(z - \frac{1}{z})\right\}\frac{1}{2}(z - \frac{1}{z}) = \sum_{n=-\infty}^{\infty} z^n J_n'(x)$$

$$\Rightarrow \quad \frac{1}{2}(z - \frac{1}{z}) \sum_{n=-\infty}^{\infty} z^n J_n(x) = \sum_{n=-\infty}^{\infty} z^n J_n'(x)$$

$$\Rightarrow \quad \frac{1}{2} \sum_{n=-\infty}^{\infty} z^{n+1} J_n(x) - \frac{1}{2} \sum_{n=-\infty}^{\infty} z^{n-1} J_n(x) = \sum_{n=-\infty}^{\infty} z^n J_n'(x)$$

Equating the coefficient of z^n both side we get

$$\frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) = J_n'(x) \Rightarrow 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

5.6 Worked out Examples

Example 5.4 Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Solution: From the recurrence relation-III, we have

$$J_{n+1}(x) = \tfrac{2n}{x} J_n(x) - J_{n-1}(x)$$

Putting $n = 4, 3, 2, 1$ we get

$$\begin{aligned}
J_5(x) &= \frac{8}{x}J_4(x) - J_3(x), \qquad J_4(x) = \frac{6}{x}J_3(x) - J_2(x) \\
J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) \qquad \text{and } J_2(x) = \frac{2}{x}J_1(x) - J_0(x). \\
\text{Now } J_5(x) &= \frac{8}{x}J_4(x) - J_3(x) = \frac{8}{x}\Big(\frac{6}{x}J_3(x) - J_2(x)\Big) - J_3(x) = \frac{48}{x^2}J_3(x) - \frac{8}{x}J_2(x) - J_3(x) \\
&= \Big(\frac{48}{x^2} - 1\Big)J_3(x) - \frac{8}{x}J_2(x) = \Big(\frac{48}{x^2} - 1\Big)\Big(\frac{4}{x}J_2(x) - J_1(x)\Big) - \frac{8}{x}J_2(x) \\
&= \Big(\frac{192}{x^3} - \frac{4}{x}\Big)J_2(x) - \Big(\frac{48}{x^2} - 1\Big)J_1(x) - \frac{8}{x}J_2(x) = \Big(\frac{192}{x^3} - \frac{12}{x}\Big)J_2(x) - \Big(\frac{48}{x^2} - 1\Big)J_1(x) \\
&= \Big(\frac{192}{x^3} - \frac{12}{x}\Big)\Big(\frac{2}{x}J_1(x) - J_0(x)\Big) - \Big(\frac{48}{x^2} - 1\Big)J_1(x) = \Big(\frac{192}{x^3} - \frac{12}{x}\Big)\frac{2}{x}J_1(x) - \Big(\frac{192}{x^3} - \frac{12}{x}\Big)J_0(x) \\
&- \Big(\frac{48}{x^2} - 1\Big)J_1(x) = \Big(\frac{384}{x^4} - \frac{72}{x^2} + 1\Big)J_1(x) - \Big(\frac{192}{x^3} - \frac{12}{x}\Big)J_0(x)
\end{aligned}$$

Example 5.5 Show that

$$(a)\ J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x \qquad (b)\ J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x \qquad (c)\ \{J_{-\frac{1}{2}}(x)\}^2 + \{J_{\frac{1}{2}}(x)\}^2 = \frac{2}{\pi x}$$

Proof: From the Bessel's function of first kind, we have

$$J_n(x) = \frac{1}{2^n\Gamma(n+1)}x^n\left[1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4.8(1+n)(2+n)} - \cdots\right]$$

$$\Rightarrow J_n(x) = \frac{1}{2^n\Gamma(n+1)}x^n\left[1 - \frac{x^2}{2(2+2n)} + \frac{x^4}{2.4(2+2n)(4+2n)} - \cdots\right] \tag{5.14}$$

(a) Putting $n = -\frac{1}{2}$ in (5.14), we have

$$\begin{aligned}
J_{-\frac{1}{2}}(x) &= \frac{1}{2^{-\frac{1}{2}}\Gamma(-\frac{1}{2}+1)}x^{-\frac{1}{2}}\left[1 - \frac{x^2}{2.1} + \frac{x^4}{2.4.3.1} - \cdots\right] \\
&= \frac{1}{2^{-\frac{1}{2}}\Gamma(\frac{1}{2})}x^{-\frac{1}{2}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] = \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] \ [\text{ as } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}] \\
&= \sqrt{\frac{2}{\pi x}}\cos x
\end{aligned}$$

(b) Putting $n = \frac{1}{2}$ in (5.14), we have

$$\begin{aligned}
J_{\frac{1}{2}}(x) &= \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{1}{2}+1)}x^{\frac{1}{2}}\left[1 - \frac{x^2}{1.2.3} + \frac{x^4}{1.2.4.3.5} - \cdots\right] \\
&= \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{3}{2})}x^{\frac{1}{2}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right] = \frac{1}{2^{\frac{1}{2}}\frac{1}{2}\Gamma(\frac{1}{2})}x^{\frac{1}{2}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right] \ [\text{ as } \Gamma(n+1) = n\Gamma(n)] \\
&= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] = \sqrt{\frac{2}{\pi x}}\sin x, \ [\text{ as } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}].
\end{aligned}$$

(c) Squaring and adding (a) and (b), we have $\{J_{-\frac{1}{2}}(x)\}^2 + \{J_{\frac{1}{2}}(x)\}^2 = \frac{2}{\pi x}$.

Example 5.6 Prove that $(a)\ J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}(-\frac{\cos x}{x} - \sin x)$ (b)$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}(\frac{\sin x}{x} - \cos x)$.

Solution: From example 5.5, we have

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}cosx \tag{5.15}$$

$$\text{and } J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}sinx \tag{5.16}$$

Also from recurrence relation IV, we have

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \tag{5.17}$$

(a) Putting $n = -\frac{1}{2}$ in (5.17), we get

$$J_{-\frac{3}{2}}(x) + J_{\frac{1}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) \Rightarrow J_{-\frac{3}{2}}(x) = -J_{\frac{1}{2}}(x) - \frac{1}{x}J_{-\frac{1}{2}}(x)$$
$$= \sqrt{\frac{2}{\pi x}}(-\frac{\cos x}{x} - \sin x) \text{ [Using (5.15) and (5.16)]}$$

(b)Putting $n = \frac{1}{2}$ in (5.17), we get

$$J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x)$$
$$\Rightarrow J_{\frac{3}{2}}(x) = -J_{-\frac{1}{2}}(x) + \frac{1}{x}J_{\frac{1}{2}}(x)$$
$$= \sqrt{\frac{2}{\pi x}}(-\frac{\sin x}{x} - \cos x) \text{ [Using (5.15) and (5.16)]}$$

Example 5.7 Show that

$$\text{(i) } \cos x = J_0(x) - 2J_1(x) + 2J_4(x) - \cdots$$
$$\text{(ii) } \sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \cdots$$

Solution: From generating function of Bessel's function, we have

$$\exp\left\{\frac{x}{2}(z - \frac{1}{z})\right\} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$
$$= \cdots + z^{-3}J_{-3}(x) + z^{-2}J_{-2}(x) + z^{-1}J_{-1}(x) + J_0(x) + z^1 J_1(x) + z^2 J_2(x) + z^3 J_3(x) + \cdots$$
$$= J_0(x) + (z - \frac{1}{z})J_1(x) + (z^2 + \frac{1}{z^2})J_2(x) + (z^3 - \frac{1}{z^3})J_3(x) + \cdots \quad \text{[Using } J_{-n}(x) = (-1)^n J_n(x)\text{]}$$

Putting $z = e^{i\theta}$, we have

$$exp\left\{\frac{x}{2}(e^{i\theta} - e^{-i\theta})\right\} = J_0(x) + (e^{i\theta} - e^{-i\theta})J_1(x) + (e^{2i\theta} + e^{-2i\theta})J_2(x) + (e^{3i\theta} - e^{-3i\theta})J_3(x) + \cdots$$
$$\Rightarrow e^{ix\sin\theta} = J_0(x) + (2i\sin\theta)J_1(x) + (2\cos 2\theta)J_2 + (2i\sin 3\theta)J_3(x) + \cdots$$
$$\Rightarrow \cos(x\sin\theta) + i\sin(x\sin\theta) = (J_0 + 2\cos 2\theta J_2 + 2\cos 4\theta J_4 + \cdots)$$
$$+ \quad i(2\sin\theta J_1 + 2\sin 3\theta J_3 + \cdots)$$

Equating the real and imaginary parts, we have

$$\cos(x\sin\theta) = J_0 + 2\cos 2\theta J_2 + 2\cos 4\theta J_4 + \cdots)$$
$$\sin(x\sin\theta) = 2\sin\theta J_1 + 2\sin 3\theta J_3 + \cdots$$

Putting $\theta = \frac{\pi}{2}$, we have

$$\cos x = J_0(x) - 2J_1(x) + 2J_4(x) - \cdots$$
$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \cdots$$

Example 5.8 Show that $\int_0^x x^{-n}J_{n+1}(x)dx = \frac{1}{2^n\Gamma(n+1)} - x^{-n}J_n(x)$, $n > 1$.

Solution: From recurrence relation II, we have

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x)$$

Integrating both sides with respect to 'x' between the limits 0 and x, we get

$$\left[x^{-n}J_n(x)\right]_0^x = -\int_0^x x^{-n}J_{n+1}(x)dx$$

$$\Rightarrow \quad x^{-n}J_n(x) - \lim_{x\to 0}\frac{J_n(x)}{x^n} = -\int_0^x x^{-n}J_{n+1}(x)dx$$

But $\lim_{x\to 0}\frac{J_n(x)}{x^n} = \frac{1}{x^n}\frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{4(1+n)} + \cdots\right] = \frac{1}{2^n\Gamma(n+1)}$

Hence $\int_0^x x^{-n}J_{n+1}(x)dx = \frac{1}{2^n\Gamma(n+1)} - x^{-n}J_n(x), n > 1.$

Example 5.9 Express $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Solution: From the recurrence relation, we have

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Putting $n = 2, 1$ we get

$$\begin{aligned} J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) \text{ and } J_2(x) = \frac{2}{x}J_1(x) - J_0(x). \\ \text{Now } J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) = \frac{4}{x}\Big(\frac{2}{x}J_1(x) - J_0(x)\Big) - J_1(x) \\ &= \frac{8}{x^2}J_1(x) - J_1(x) - \frac{4}{x}J_0(x) = \Big(\frac{8}{x^2} - 1\Big)J_1(x) - \frac{4}{x}J_0(x). \end{aligned}$$

Example 5.10 Find the value of $J_1(x)$.

Solution: We know that

$$J_n(x) = \frac{1}{2^n\Gamma(n+1)}x^n[1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4.8(1+n)(2+n)} - \cdots]$$

Putting $n = 1$, we get

$$\begin{aligned} J_1(x) &= \frac{1}{2\Gamma(1+1)}x[1 - \frac{x^2}{4(1+1)} + \frac{x^4}{4.8(1+1)(2+1)} - \cdots] \\ J_1(x) &= \frac{x}{2}[1 - \frac{x^2}{8} + \frac{x^4}{192} - \cdots] \end{aligned}$$

Example 5.11 Write down the Bessel's equation of order 2.

Solution: The differential equation of the form

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

is called Bessel equation of order n. Putting $n = 2$, we get $x^2y'' + xy' + (x^2 - 4)y = 0$.

Example 5.12 Prove that $J_n(-x) = (-1)^nJ_n(x)$ using generating function.

Solution: We have

$$exp\{\tfrac{x}{2}(z-\tfrac{1}{z})\} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

Replacing x by $-x$ we get

$$\sum_{n=-\infty}^{\infty} z^n J_n(-x) = exp\{\frac{-x}{2}(z-\frac{1}{z})\} = exp\{\frac{x}{2}(-z-\frac{1}{-z})\}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} z^n J_n(-x) = \sum_{n=-\infty}^{\infty} (-z)^n J_n(x)$$

Equating the coefficients of z^n from both sides, we get $J_n(-x) = (-1)^n J_n(x)$.

5.7 Multiple Choice Questions(MCQ)

1. The general solution to the differential equation

$$x^2\frac{d^2x}{dy^2} + x\frac{dy}{dx} + \Big(4x^2 - \frac{5}{25}\Big)y = 0 \text{ is} \qquad \textbf{GATE(MA)} - \textbf{2014}$$

A) $y(x) = \alpha J_{\frac{3}{5}}(2x) + \beta J_{-\frac{3}{5}}(2x)$ B) $y(x) = \alpha J_{\frac{3}{10}}(x) + \beta J_{-\frac{3}{10}}(x)$
C) $y(x) = \alpha J_{\frac{3}{5}}(x) + \beta J_{-\frac{3}{5}}(x)$ D) $y(x) = \alpha J_{\frac{3}{10}}(2x) + \beta J_{-\frac{3}{10}}(2x)$
Ans. (A)

2. It is known that Bessel function $J_n(x)$, $n \geq 0$, satisfy the identity $e^{\frac{x}{2}(t-\frac{1}{t})} = J_0(x) + \sum\limits_{n=1}^{\infty} J_n(x)\Big(t^n + \frac{(-1)^n}{t^n}\Big)$ for all $t > 0$, and $x \in \Re$. The value of $J_0(\frac{\pi}{3}) + 2\sum\limits_{n=1}^{\infty} J_{2n}(\frac{\pi}{3})$ is equal to **GATE(MA)-2015**
(A) 2 (B) 1 (C) 3 (D) 0
Ans. (B)

Hint. We have put $t = 1$, we get $1 = J_0(x) + 2\sum\limits_{n=1}^{\infty} J_{2n}(x)$, $x \in \Re$. Then replacing x by $\frac{\pi}{3}$, we obtain $J_0(\frac{\pi}{3}) + 2\sum\limits_{n=1}^{\infty} J_{2n}(\frac{\pi}{3}) = 1$.

3. If $J_n(x)$ and $Y_n(x)$ denote Bessel functions of order n of the first and second kind, then the general solution of the differential equation $x\frac{d^2x}{dy^2} - x\frac{dy}{dx} + xy = 0$ is **GATE(MA)-2005**
A) $y(x) = \alpha x J_1(x) + \beta x Y_1(x)$ B) $y(x) = \alpha J_0(x) + \beta Y_0(x)$
C) $y(x) = \alpha J_1(x) + \beta Y_1(x)$ D) $y(x) = \alpha x J_0(x) + \beta x Y_0(x)$
Ans. A)

5.8 Review Exercises

1. Show that $\int_0^x x^3 J_0(x)dx = x^3 J_1(x) - 2x^2 J_2(x)$ **GATE(MA)-2003**
2. Prove that $\frac{d}{dx}\{xJ_1(x)\} = xJ_0(x)$.
3. Prove that $\frac{d}{dx}\{J_0(x)\} = -J_1(x)$.

4. Prove that $J_{-\frac{5}{2}} = (\frac{2}{\pi x})^{\frac{1}{2}}[\frac{3-x^2}{x^2} cosx + \frac{3}{x} sinx]$.
5. Prove that $J_{\frac{5}{2}} = (\frac{2}{\pi x})^{\frac{1}{2}}[\frac{3-x^2}{x^2} sinx - \frac{3}{x} cosx]$.
6. Show that $\int\limits_0^{\frac{\pi}{2}} \sqrt{\pi x} J_{\frac{1}{2}}(x)dx = 1$.
7. Express in $J_3(x)$ terms of $J_0(x)$ and $j_1(x)$. [Ans: $J_3(x) = \frac{8-x^2}{x^2} J_1(x) - \frac{4}{x} J_0(x)$]
8. Show that $J_n(x) = 0$ has no repeated roots except at $x = 0$.
9. Prove that $\int\limits_a^b xJ_0(ax)dx = \frac{b}{a}$.
10. Prove that $J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = 1$
11. Show that all the roots of $J_n(x)$ are real.
12. Evaluate $\int\limits_{-1}^{1} x^4 J_1(x)dx$. [$Ans : x^4 J_2 - 2x^3 J_3 + c$.]
13. Establish the relation $2J_n'(x) = J_{n-1}(x) + J_{n+1}(x)$ the Bessel's function $J_n(x)$. Hence deduce that
$$J_n''(x) = \frac{1}{4}[J_{n-1}(x) - 2J_n(x) + J_{n+1}(x)]$$
14. Prove that $\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2(\frac{n}{x} J_n^2 - \frac{n+1}{x} J_{n+1}^2)$.
15. If $a > 0$, prove that $\int\limits_0^{\infty} e^{-ax} J_0(bx)dx = \frac{1}{\sqrt{a^2+b^2}}$.
16. Show that $\int\limits_0^{\infty} x^{-n} J_{n+1}(x)dx = \frac{1}{2^n \Gamma(n+1)}$, $n > -\frac{1}{2}$.
17. Prove that $\int J_0(x) \sin x dx = xJ_0 \sin x - xJ_1(x) \cos x$.
18. Prove that $J_{n+3} + J_{n+5} = \frac{2}{x}(n + 4)J_{n+4}$
19. Show that $x \sin x = 2\Big(2^2 J_2 - 4^2 J_4 + 6^2 J_6 - \cdots\Big)$.
20. Show that $x \cos x = 2\Big(1^2 J_1 - 3^2 J_3 + 5^2 J_5 - \cdots\Big)$.

Bibliography

[1] Birkhoff, G. and Rota, G.C. (1989). Ordinary Differential Equations (4th ed.), Wiley, New York.

[2] Bernstein, D.L., (1951). Existence Theorems in Partial Differential Equations (AM-23), Annals of Mathematics Studies, Princeton University Press.

[3] Boyce, W. E. and DiPrima, R. C. (2001). Elementary Differential Equations and Boundary Value Problems, John Wiley & Sons, Inc. New York.

[4] Chatterjee, D., (2000). Integral Calculus and Differential Equations, Tata McGraw-Hill Publishing Company Limited, New Delhi.

[5] Coddington, E. A.(1961). An Introduction to Ordinary Differential Equations, Dover Publications, Inc., New York.

[6] Debnath, L. and Bhatta, D. (2006). Integral Transforms and Their Applications, Chapman & Hall/CRC Taylor & Francis Group.

[7] Jana, D. K. and Maity, K. (2013). Advance Engineering Numerical Methods, Macmillan Publisher India Ltd.

[8] Jordan, D. W. and Smith, P. (2009). Nonlinear Ordinary Differential Equations, Oxford University Press, Oxford.

[9] Jain, M. K., Iyengar, S.R.K. and Jain, R.K. (1984). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi.

[10] Ghosh, R. K. and Maity, K. C.(2011). An Introduction to Differential Equations (9th Edition) New Central Book Agency (P) Ltd., Kolkata.

[11] Khatua, D. and Maity, K.(2016). "Stability Analysis of a Dynamical System", (edited by Maity, K.) in "Bio-mathematical Modeling under Uncertain Environment". NAROSA, New Delhi.

[12] Liao, X., Wang, L. and Yu, P. (2007). Stability of a dynamical system. Elsevier.

[13] Liu, W.M. (1994). Criterion of Hopf bifurcation without using eigenvalues. J. Math. Anal. Appl, 182 : 250.

[14] Ma, Z.,Wang, Y. and Jiang, G. (2012). Bifurcation analysis of a linear Hamiltonian system with two kinds of impulsive control. Nonlinear Dyn., 70 : 2367-2374.

[15] Maity, K. (2016). Bio-Mathematical Modelling Under Uncertain Environment(Edited Book), Narosa, New Delhi.

[16] Makinde, O.D. (2007). Solving ratio-dependent predatorprey system with constant effort harvesting using Adomain decomposition method. Appl. Math.Comput, 187 : 17-22.

[17] Mazandarani, M., Najariyan, M.,(2014). A note on ”A class of linear differential dynamical systems with fuzzy initial condition”, Fuzzy Sets and Systems, Article in press.

[18] Mondal, N.(2015). Differential Equations (Ordinary and Partial), Books and Applied (P) Ltd, Kolkata.

[19] Mondal, S. K.,(2016). ”An Overview on Stability Analysis of a Nonlinear System and its Application in Prey-Predator Model”, (edited by Maity, K.) in ”Bio-Mathematical Modelling Under Uncertain Environment”, Narosa, New Delhi.

[20] Morley, M. (1965). Categoricity in Power. Transactions of the American Mathematical Society (American Mathematical Society), 114 (2): 514-538.

[21] Mukherjee, A., Bej, N. K. (2009). Ordinary and Partial Differential Equations, Shreedhar Prakashani, Kolkata.

[22] Xie, W. C.(2010). Differential Equations for Engineers, Cambridge University Press, Cambridge.

[23] Parks, P.C. and Hahn, V.,(1992). Stability Theory, Prentics Hall, New York.

[24] Perko, L. (1991). Differential Equations and Dynamical Systems. Springer.

[25] Raisinghania, M,D. (2011). Ordinary and Partial Differential Equations, S. Chand and Company Ltd., New Delhi.

[26] Rao, K. S.(2009). Introduction to Partial Differential Equations, Phi Learning Private Limited, New Delhi.

[27] Ross, S. L.(2004). Differential Equations, John Wiley & Sons(Asia) Pre. Ltd., Singapore.

[28] Shelah, S. (1969). Stable theories. Israel J. Math., 7 (3): 187-202.

[29] Shelah, S. (1990). Classification theory and the number of nonisomorphic models. Studies in Logic and the Foundations of Mathematics (2nd ed.), Elsevier.

[30] Somasundaram, D. (2010). Ordinary Differential Equations-A first cource, Narosa Publishing House.

[31] Spiegel, M.R. (1965). Theory and Problems of Laplace Transforms, Schaum’s Outline Series, MacGraw-Hill Book Company, Inc.

[32] Weinberger, H. (1995). A First Course in Partial Differential Equations, Wiley, New York.

[33] Xiao, D., Li, W. and Han, W. (2006). Dynamics in a ratio dependent predator prey modelwith predator harvesting. J. Math.Anal. Appl., 324(1): 4-29.

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