INTRODUCTION TO
DIFFERENTIAL EQUATIONS

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To my beloved Daughters

Samadrita & Somdatta

Preface

With the remarkable advancement in various branches of science, engineering and technology, today more than ever before, the study of differential equations has become essential. For, to have an exhaustive understanding of subjects like physics, mathematical biology, chemical science, mechanics, fluid dynamics, heat transfer, aerodynamics, electricity, waves and electromagnetic, the knowledge of finding solution to differential equations is absolutely necessary. These differential equations may be ordinary or partial. Finding and interpreting their solutions are at the heart of applied mathematics. A thorough introduction to differential equations is therefore a necessary part of the education of any applied mathematician, and this book is aimed at building up skills in this area.

This book on ordinary / partial differential equations is the outcome of a series of lectures delivered by me, over several years, to the undergraduate or postgraduate students of Mathematics at various institution. My principal objective of the book is to present the material in such a way that would immediately make sense to a beginning student. In this respect, the book is written to acquaint the reader in a logical order with various well-known mathematical techniques in differential equations. Besides, interesting examples solving JAM / GATE / NET / IAS / NBHM/TIFR/SSC questions are provided in almost every chapter which strongly stimulate and help the students for their preparation of those examinations from graduate level.

Organization of the book
The book has been organized in a logical order and the topics are discussed in a systematic manner. It has comprising 21 chapters altogether. In the chapter **??**, the fundamental concept of differential equations including autonomous/ non-autonomous and linear / non-linear differential equations has been explained. The order and degree of the ordinary differential equations (ODEs) and partial differential equations(PDEs) are also mentioned. The chapter **??** are concerned the first order and first degree ODEs. It is also written in a progressive manner, with the aim of developing a deeper understanding of ordinary differential equations, including conditions for the existence and uniqueness of solutions. In chapter **??** the first order and higher degree ODEs are illustrated with sufficient examples. The chapter **??** is concerned with the higher order and first degree ODEs. Several methods, like method of undetermined coefficients, variation of parameters and Cauchy-Euler equations are also introduced in this chapter. In chapter **??**, second order initial value problems, boundary value problems and Eigenvalue problems with Sturm-Liouville problems are expressed with proper examples. Simultaneous linear differential equations are studied in chapter 1. It is also written in a progressive manner with the aim of developing some alternative methods. In chapter **??**, the equilibria, stability

and phase plots of linear / nonlinear differential equations are also illustrated by including numerical solutions and graphs produced using Mathematica version 9 in a progressive manner. The geometric and physical application of ODEs are illustrated in chapter **??**. The chapter **??** is presented the Total (Pfaffian) Differential Equations. In chapter **??**, numerical solutions of differential equations are added with proper examples. Further, I discuss Fourier transform in chapter **??**, Laplace transformation in chapter **??**, Inverse Laplace transformation in chapter **??**. Moreover, series solution techniques of ODEs are presented with Frobenius method in chapter **??**, Legendre function and Rodrigue formula in Chapter **??**, Chebyshev functions in chapter **??**, Bessel functions in chapter **??** and more special functions for Hypergeometric, Hermite and Laguerre in chapter **??** in detail.

Besides, the partial differential equations are presented in chapter **??**. In the said chapter, the classification of linear, second order partial differential equations emphasizing the reasons why the canonical examples of elliptic, parabolic and hyperbolic equations, namely Laplace's equation, the diffusion equation and the wave equation have the properties that they do has been discussed. Chapter **??** is concerned with Green's function. In chapter **??**, the application of differential equations are developed in a progressive manner. Also all chapters are concerned with sufficient examples. In addition, there is also a set of exercises at the end of each chapter to reinforce the skills of the students.

Moreover it gives the author great pleasure to inform the reader that the **second edition** of the book has been improved, well -organized, enlarged and made up-to-date as per latest UGC - CBSC syllabus. The following significant changes have been made in the second edition:

- Almost all the chapters have been rewritten in such a way that the reader will not find any difficulty in understanding the subject matter.
- Errors , omissions and logical mistakes of the previous edition have been corrected.
- The exercises of all chapters of the previous edition have been improved, enlarged and well-organized.
- Two new chapters like Green's Functions and Application of Differential Equations have been added in the present edition.
- More solved examples have been added so that the reader may gain confidence in the techniques of solving problems.
- References to the latest papers of various university, IIT-JAM, GATE, and CSIR-UGC(NET) have been provided in almost every chapters which strongly help the students for their preparation of those examinations from graduate label.

In view of the above mentioned features it is expected that this new edition will appreciate and be well prepared to use the wonderful subject of differential equations.

Aim and Scope
When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results of the modelling process is a system of ordinary or partial differential equations. Finding and interpreting the solutions of these differential equations

is therefore a central part of applied mathematics, Physics and a thorough understanding of differential equations is essential for any applied mathematician and physicist. The aim of this book is to develop the required skills on the part of the reader. The book will thus appeal to undergraduates/postgraduates in Mathematics, but would also be of use to physicists and engineers. There are many worked examples based on interesting real-world problems. A large selection of examples / exercises including JAM/NET/GATE questions is provided to strongly stimulate and help the students for their preparation of those examinations from graduate level. The coverage is broad, ranging from basic ODE , PDE to second order ODE's including Bifurcation theory, Sturm-Liouville theory, Fourier Transformation, Laplace Transformation, Green's function and existence and uniqueness theory, through to techniques for nonlinear differential equations including stability methods. Therefore, it may be used in research organization or scientific lab.

Significant features of the book

- A complete course of differential Equations
- Perfect for self-study and class room
- Useful for beginners as well as experts
- More than 650 worked out examples
- Large number of exercises
- More than 700 multiple choice questions with answers
- Suitable for New UGC-CBSC syllabus of ODE & PDE
- Suitable for GATE, NET, NBHM, TIFR, JAM, JEST, IAS, SSC examinations.

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I shall feel great to receive constructive criticisms through email for the improvement of the book from the experts as well as the learners.

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Contents

Chapter 1

Simultaneous Linear Differential Equations

1.1 Introduction

We have only provided the definition of system of ODEs in section **??** in chapter **??**. Excepted this, in all the previous chapters, we have discussed only those differential equations which contain one independent variable and one dependent variable. In this chapter we shall consider the linear differential equation with more than one dependent variable depending on one independent variable. Such system of differential equation is called **simultaneous linear differential equations**. Here, we discussed also the methods of solution of those differential equations. Generally two types (Type-I and Type-II) of simultaneous equations are considered:

1.2 Simultaneous Linear Differential Equations of Type-I

1.2.1 Simultaneous Linear Differential Equations with constant coefficients of Type-I

The system of n linear simultaneous ordinary differential equations with constant coefficients of Type-I is the form of

$$\begin{aligned}
&\phi_{11}(D)x_1 + \phi_{12}(D)x_2 + \cdots + \phi_{1n}(D)x_n = f_1(t)\\
&\phi_{21}(D)x_1 + \phi_{22}(D)x_2 + \cdots + \phi_{2n}(D)x_n = f_2(t)\\
&\vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots\\
&\phi_{n1}(D)x_1 + \phi_{n2}(D)x_2 + \cdots + \phi_{nn}(D)x_n = f_n(t)
\end{aligned}$$

where x_1, $x_2, \cdots, x_n$ are the dependent variables dependent on t (independent variable) and $\phi_{ij}(D)$, $(i, j = 1, 2, \cdots, n)$ are all rational function of $D \equiv \frac{d}{dt}$ with constant coefficients and $f_i(t)$, $(i = 1, 2, \cdots, n)$.

1.2.2 Simultaneous Linear Differential Equations with variable coefficients of Type-I

The general non-homogeneous, first order linear system with variable coefficients of n dimensions is

$$\dot{\mathbf{x}} = \mathbf{A}(t)x + \mathbf{f}(t),$$

where $A(t)$ is an $n\times n$ matrix whose elements a_{ij} are functions of time and $\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$ and $\mathbf{f}(t) = [f_1(t), f_2(t), \cdots, f_n(t)]^T$ are the column vectors of the n variables.

1.3 Simultaneous Linear Differential Equations of Type-II

Another type of linear simultaneous equations is

$$\begin{aligned} & P_1dx + Q_1dy + R_1dz = 0 \\ \text{and} \qquad & P_2dx + Q_2dy + R_2dz = 0 \end{aligned}$$

where P_1, Q_1, R_1, P_2, Q_2, R_2 are functions of x, y and z.
Now by cross-multiplication, we get

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1}$$
$$\text{which is of the form } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where P, Q and R are functions of x, y, z.

1.4 Lipschitz (Cauchy-Lipschitz) condition

A vector-valued function $\mathbf{f}$ defined for $(t, \mathbf{x})$ in some set D (t real, $\mathbf{x}$ in $\Re^n$) is said to be continuous on D. The function $\mathbf{f}$ satisfies a Lipschitz condition on D if there exists a constant $\lambda > 0$ such that

$$||\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})|| \leq \lambda||\mathbf{x} - \mathbf{y}||$$

for all $(t, \mathbf{x})$, $(t, \mathbf{y})$ in D where $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Re^n$, $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \Re^n$. The constant λ is known as Lipschitz constant for the corresponding function $\mathbf{f}$ on D.

Example 1.1 Show that $\mathbf{f}(t, \mathbf{x}) = (3t + 2x_1, x_1 - x_2)$ on $S : \{|t| < \infty, |x| < \infty\}$ satisfying a Lipschitz condition.

Proof.: Here

$$\begin{aligned} ||\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})|| &= ||2(x_1 - y_1), (x_1 - y_1) - (x_2 - y_2)|| \\ &= 2|x_1 - y_1| + ||(x_1 - y_1) - (x_2 - y_2)|| \\ &= 2|x_1 - y_1| + |x_1 - y_1| + |x_2 - y_2| \\ &\leq 3||\mathbf{x} - \mathbf{y}|| \end{aligned}$$

So, $||\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{y})|| \le \lambda||\mathbf{x}-\mathbf{y}||$ with $\lambda = 3$ for $(t,\mathbf{x})$ on S. Therefore, the given function $\mathbf{f}$ satisfy the Lipschitz condition on S.

Theorem 1.1 Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be in $\Re^n$. Also suppose $\mathbf{f}$ be a vector-valued function defined for $(t,\mathbf{x})$ on a set D of the form

$$|t - t_0| \le a, \quad ||\mathbf{x} - \mathbf{x_0}|| \le b, \quad (a,\, b > 0),$$

or of the form

$$|t - t_0| \le a, \quad ||\mathbf{x}|| < \infty, \quad (a > 0).$$

If $\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial x_k}$, $(k = 1, 2, \cdots, n)$ exists is continuous on D and there is a constant $\lambda > 0$ such that

$$||\frac{\partial f(t,\mathbf{x})}{\partial x_k}|| \le \lambda, \ (k = 1, 2, \cdots, n), \ \forall (t,\mathbf{x}) \ \text{in} \ D. \tag{1.1}$$

Then f satisfies a Lipschitz condition on D with Lipschitz constant λ.

Proof: Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be in $\Re^n$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ be in $\Re^n$. Let $(t,\mathbf{x})$, $(t,\mathbf{y})$ be two fixed points in D and define the vector-valued function $\mathbf{F}$ for real s, $0 \le s \le 1$, by

$$\mathbf{F}(s) = \mathbf{f}(t, \mathbf{y} + s(\mathbf{x} - \mathbf{y})), \ (0 \le s \le 1).$$

This is a well-defined function since the points $(t, \mathbf{y} + s(\mathbf{x}-\mathbf{y}))$ are in D for $0 \le s \le 1$. Clearly $|t - t_0| \le a$, and if $||\mathbf{x} - \mathbf{x_0}|| \le b$, $||\mathbf{y} - \mathbf{y_0}|| \le b$, then

$$\begin{aligned} ||\mathbf{y} + s(\mathbf{x}-\mathbf{y}) - \mathbf{x_0}|| &= ||(1-s)(\mathbf{y} - \mathbf{x_0}) + s(\mathbf{x} - \mathbf{x_0})|| \\ &\le (1-s)||\mathbf{y} - \mathbf{x_0}|| + s||\mathbf{x} - \mathbf{x_0}|| \\ &\le (1-s)b + sb = b. \end{aligned}$$

If $||\mathbf{x}|| < \infty$, $||\mathbf{y}|| < \infty$, then

$$||\mathbf{y} + s(\mathbf{x}-\mathbf{y})|| \le (1-s)||\mathbf{y}|| + s||\mathbf{x}|| \le ||\mathbf{x}|| + ||\mathbf{y}|| < \infty.$$

We now have

$$\mathbf{F}'(s) = (x_1 - y_1)\frac{\partial \mathbf{f}}{\partial x_1}(t, \mathbf{y} + s(\mathbf{x}-\mathbf{y})) + \cdots + (x_n - y_n)\frac{\partial \mathbf{f}}{\partial x_n}(t, \mathbf{y} + s(\mathbf{x}-\mathbf{y})),$$

so that

$$||\mathbf{F}'(s)|| \le ||(x_1 - y_1)||||\frac{\partial \mathbf{f}}{\partial x_1}(t, \mathbf{y} + s(\mathbf{x}-\mathbf{y}))|| + \cdots + ||(x_n - y_n)||||\frac{\partial \mathbf{f}}{\partial x_n}(t, \mathbf{y} + s(\mathbf{x}-\mathbf{y}))||.$$

Then using (1.1), we have that

$$\mathbf{F}'(s) \le \lambda||\mathbf{x} - \mathbf{y}||, \ (0 \le s \le 1).$$

Thus, since

$$||\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{y})|| = ||\mathbf{F}(1) - \mathbf{F}(0)|| = ||\int_0^1 \mathbf{F}'(s)ds|| \le \int_0^1 ||\mathbf{F}'(s)||ds \le \lambda||\mathbf{x} - \mathbf{y}||.$$

So we have,

$$\|\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{y})\| \leq \lambda \|\mathbf{x} - \mathbf{y}\|,$$

which is the theorem.

Example 1.2 Show that $\mathbf{f}(t,\mathbf{x}) = (3t + 2x_1,\ x_1 - x_2)$ on $S:\ \{|t| < \infty,\ \ \|\mathbf{x}\| < \infty\}$ satisfying a Lipschitz condition.

Proof.: Here

$$\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial x_1} = (2,1),\quad \frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial x_2} = (0,-1)$$

and hence

$$\|\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial x_1}\| = 3,\quad \|\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial x_2}\| = 1.$$

Using the Theorem 1.1, we have $\mathbf{f}$ satisfies a Lipschitz condition on S with a Lipschitz constant $\lambda = 3$.

Example 1.3 Consider the system of two equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + g_i(t),\ \ i = 1, 2, \cdots, n,$$

where $a_{ij},\ (i,\ j = 1, 2, \cdots, n)$ are constants and $g_i,\ i = 1, 2, \cdots, n$ are continuous in $\mathfrak{R}$. If the system is written in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t,\mathbf{x}),$$

then show that the $\mathbf{f}$ satisfies a Lipschitz condition for all $(t,\mathbf{x})$ where t is real and $\mathbf{x}$ is in $\mathfrak{R}^n$.

Proof.: Here $\mathbf{f}(t,\mathbf{x}) = (f_1(t,\mathbf{x}), f_2(t,\mathbf{x}), \cdots, f_n(t,\mathbf{x}))$ with $f_i(t,\mathbf{x}) = \sum_{j=1}^{n} a_{ij}x_j + g_i(t),\ i = 1, 2, \cdots, n$. Therefore,

$$\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial x_i} = (a_{1i}, a_{2i}, \cdots, a_{ni})$$

and hence

$$\|\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial x_i}\| = \sum_{j=1}^{n} |a_{ji}|.$$

Using the Theorem 1.1, we have $\mathbf{f}$ satisfies a Lipschitz condition for all $(t,\mathbf{x})$ where t is real and $\mathbf{x}$ is in $\mathfrak{R}^n$.

Theorem 1.2 (**Local Existence and Uniqueness**)
Let $\mathbf{f}$ be a continuous vector-valued function defined on

$$R : |t - t_0| \leq a, \ \ ||\mathbf{x} - \mathbf{x_0}|| \leq b, \ \ (a, b > 0),$$

and suppose $\mathbf{f}$ satisfies a Lipschitz condition on R. If M be a positive constant such that

$$||\mathbf{f}(t, \mathbf{x})|| \leq M$$

for all $(t, \mathbf{x})$ in R, the successive approximations $\{\phi_k\}$, $(k = 0, 1, 2, \cdots)$, given by

$$\phi_{\mathbf{0}}(t) = \mathbf{x_0}, \ \ \phi_{\mathbf{k+1}}(t) = \mathbf{x_0} + \int_{t_0}^{t} \mathbf{f}(t, \phi_{\mathbf{k}}(x))dt, \ \ (k = 0, 1, 2, \cdots)$$

converge on the interval

$$I : \left\{|t - t_0| \leq \alpha = minimum\{a, \frac{b}{M}\}\right\}$$

to a unique solution ϕ of the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \ \ \mathbf{x}(t_0) = \mathbf{x_0},$$

on I.

Proof.: The proof is the same as that of Theorems **??** & **??** of Chapter **??** with x, y, f, ϕ replaced everywhere by t, $\mathbf{x}$, $\mathbf{f}$, ϕ.

Theorem 1.3 (**Non-local Existence**)
Let $\mathbf{f}$ be a continuous vector-valued function defined on

$$S : |t - t_0| \leq a, \ \ ||\mathbf{x}|| < \infty, \ \ (a > 0),$$

and satisfy there a Lipschitz condition.Then the successive approximations $\{\phi_k\}$, $(k = 0, 1, 2, \cdots)$ given by

$$\phi_{\mathbf{0}}(t) = \mathbf{x_0}, \ \ \phi_{\mathbf{k+1}}(t) = \mathbf{x_0} + \int_{t_0}^{t} \mathbf{f}(t, \phi_{\mathbf{k}}(x))dt, \ \ (k = 0, 1, 2, \cdots)$$

for the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \ \ \mathbf{x}(t_0) = \mathbf{x_0}, \ \ (||\mathbf{x_0}|| < \infty),$$

exist on $|t - t_0| \leq a$ and converge there to a solution Φ of the problem.

Proof.: The proofs carry over directly from those for Theorem **??** of Chapter **??** with x, y, f, Φ replaced everywhere by t, $\mathbf{x}$, $\mathbf{f}$, ϕ.

Theorem 1.4 (**Corollary of Non-local Existence**)
Let $\mathbf{f}$ be a continuous vector-valued function defined on

$$S : |t| < \infty, \quad \|\mathbf{x}\| < \infty,$$

and satisfies a Lipschitz condition on each "strip"

$$|t| \leq a, \quad \|\mathbf{x}\| < \infty, \ (a > 0).$$

Then every initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x_0},$$

has a solution which exists for all real x.

Proof.: The proofs carry over directly from those for Theorem **??** of Chapter **??**.

Theorem 1.5 Consider a linear system

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x} \in \mathfrak{R}^n$$

where the components $f_1, f_2, \cdots, f_n$ of $\mathbf{f}$ are given by

$$f_j(t, \mathbf{x}) = \sum_{k=1}^{n} a_{jk}(t) x_k + b_j(t), \quad (j = 1, 2, \cdots, n), \tag{1.2}$$

and the functions a_{jk}, b_j are continuous on an interval $[a, b]$ containing t_0. If $\mathbf{x_0}$ is any vector in $\mathfrak{R}^n$ there exists one and only one solution ϕ of the problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x_0}$$

on $[a, b]$.

Proof.: Here a_{jk}, b_j are continuous on an interval $[a, b]$ containing t_0, so a_{jk}, b_j are bounded on the said interval $[a, b]$ containing t_0. Therefore, there exist a positive constant λ such that

$$\sum_{j=1}^{n} |a_{jk}(t)| \leq \lambda, \quad (k = 1, 2, \cdots, n),$$

for all t satisfying $a \leq t \leq b$ containing t_0. Then from (1.60), we see that

$$\|\frac{\partial \mathbf{f}}{\partial x_k}(t, \mathbf{x})\| = \|(a_{1k}(t), a_{2k}(t), \cdots, a_{nk}(t))\| = \sum_{j=1}^{n} |a_{jk}(t)| \leq \lambda, \quad (k = 1, 2, \cdots, n).$$

Hence by Theorem 1.1, $\mathbf{f}$ satisfying a Lipschitz condition on the strip

$$S : a \leq t \leq b \ \textit{containing} \ t_0, \ \|x\| < \infty$$

with Lipschitz constant λ. Then by Theorems 1.3 & 1.2, we have the said Theorem 1.5.

1.5 Linear higher order differential equations

Consider the equation

$$\frac{d^n x(t)}{dt^n} + P_1(t)\frac{d^{n-1}x(t)}{dt^{n-1}} + P_2(t)\frac{d^{n-2}x(t)}{dt^{n-2}} + \cdots + P_n(t)x(t) = G(t), \text{ on } [a,\, b] \tag{1.3}$$

$$\text{Subject to the } n \text{ initial conditions } x(t_0) = \alpha_1,\ \frac{dx(t_0)}{dt} = \alpha_2, \cdots,\ \frac{d^{n-1}x(t_0)}{dt^{n-1}} = \alpha_n \tag{1.4}$$

where $P_1, P_2, \cdots P_n$ and G are continuous functions on $[a,\, b]$.

Theorem 1.6 Let $P_1, P_2, \cdots, P_n, G$ be continuous real valued functions on an interval $[a,\, b]$ containing a point t_0. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are any n constants, there exists one and only one solution ϕ of the equation (1.3) on $[a,\, b]$ satisfying (1.4).

Proof.: Let $\mathbf{x_0} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. Also the differential equation (1.3) with initial conditions (1.4) can also be written in the first order system

$$\frac{dx_1(t)}{dt} = x_2(t), \text{ where } x_1(t) = x(t) \tag{1.5}$$

$$\frac{dx_2(t)}{dt} = x_3(t) \tag{1.6}$$

$$\vdots \quad \vdots \quad \vdots \tag{1.7}$$

$$\frac{dx_{n-1}(t)}{dt} = x_n(t) \tag{1.8}$$

$$\frac{dx_n(t)}{dt} = -P_n(t)x_1(t) - P_{n-1}(t)x_2(t) - \cdots - P_1(t)x_n(t) + G(t) \tag{1.9}$$

Subject to the n initial conditions

$$x_1(t_0) = \alpha_1,\ x_2(t_0) = \alpha_2, \cdots,\ x_n(t_0) = \alpha_n \tag{1.10}$$

According to Theorem 1.5, there is a unique solution $\mathbf{\Phi} = (\phi_1, \phi_2, \cdots, \phi_n)$ of this system on $[a,\, b]$ satisfying

$$\phi_1(t_0) = \alpha_1,\ \phi_2(t_0) = \alpha_2,\ \cdots,\ \phi_n(t_0) = \alpha_n.$$

But since

$$\phi_2 = \phi_1^{'},\ \phi_3 = \phi_2^{'} = \phi_1^{''},\ \cdots,\ \phi_n = \phi_1^{(n-1)},$$

the function ϕ_1 is the required solution on $[a,\, b]$.

Note*: The Theorem 1.6 includes Theorem **??** of Chapter **??**.
Note:** In other words the problem of solving the initial value problem for the higher order equation (1.3) with initial condition (1.4) is equivalent to solving the initial value problem for the first order system (1.5)-(1.9) with initial condition (1.10).

Example 1.4 Find the first order simultaneous differential equations of the third order differential equation

$$\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + 12\frac{dx}{dt} - 8x = 18e^{2t}$$

with initial conditions $x(0) = c_1,\ \frac{dx(0)}{dt} = c_2,\ \frac{d^2x(0)}{dt^2} = c_3$

Solution: Write as the equivalent first order system

$$\begin{aligned}
\frac{dx}{dt} &= y, \ \frac{d^2x}{dt^2} = \frac{dy}{dt} = z, \ \frac{d^3x}{dt^3} = \frac{dz}{dt} = 8x - 12y + 6z + 18e^{2t} \\
\text{or}, \frac{dx}{dt} &= 0 \cdot x + 1 \cdot y + 0 \cdot z, \ \frac{dy}{dt} = 0 \cdot x + 0 \cdot y + 1 \cdot z, \ \frac{dz}{dt} = 8x - 12y + 6z + 18e^{2t} \\
\text{or}, \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 18e^{2t} \end{pmatrix}
\end{aligned}$$

with initial conditions

$$\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_2 \end{pmatrix}.$$

Theorem 1.7 There exists a set of n linearly independent solutions of $\dot{x}(t) = A(t)x(t)$.

Proof. The given differential equations is a system of n first order linear ordinary differential equations. We know that the $n-th$ order ordinary differential equation is equivalent to a system of n first order ordinary differential equations. By using the Theorem **??**, we have, the $n - th$ order ordinary differential equation has n linearly independent solutions. So the equivalent system of n first order ordinary differential equations has also a set of n linearly independent solutions.

Theorem 1.8 Let $\Phi_1(t), \Phi_2(t), \cdots, \Phi_n(t)$ be any set of linearly independent vector solutions of the homogeneous linear system of differential equation $\dot{x}(t) = A(t)x(t)$ on $[a, b]$. Then every solution is a linear combination of these solutions i.e $\Phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t)$ is also a solution of this homogenous linear differential equation on $[a, b]$ where $c_1, c_2, \cdots, c_n$ are n arbitrary constants.

Proof. The given differential equations is a system of n first order linear ordinary differential equations. We know that the $n-th$ order ordinary differential equation is equivalent to a system of n first order ordinary differential equations. Then the proof is entirely similar to the proof of Theorem **??**.

Definition 1.1 (Fundamental matrix) Let $\Phi_1(t), \Phi_2(t), \cdots, \Phi_n(t)$ be n linearly independent solutions of the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ on $[a, b]$. Then the matrix $\Phi(t) = \Big[\phi_1(t), \phi_2(t), \cdots, \phi_n(t)\Big] = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) & \cdots & \phi_{1n}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \cdots & \phi_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \phi_{n1}(t) & \phi_{n2}(t) & \cdots & \phi_{nn}(t) \end{pmatrix}$ is called a **fundamental matrix** of the homogenous system.

Theorem 1.9 Given any $n \times n$ solution matrix $\Phi(t) = \Big[\phi_1(t), \phi_2(t), \cdots, \phi_n(t)\Big]$ of the homogenous system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ on $[a, b]$, then either (i) for all $t \in [a, b]$, $\det\{\Phi(t)\} = 0$, or (ii) for all $t \in [a, b]$, $\det\{\Phi(t)\} \neq 0$. Case (i) occurs if and only if the solutions are linearly dependent, and case (ii) implies that $\Phi(t)$ is a fundamental matrix.

Proof. The given differential equations is a system of n first order linear ordinary differential equations. We know that the $n-th$ order ordinary differential equation is equivalent to a system of n first order ordinary differential equations. Then the proof is entirely similar to the proof of Theorems -**??** & -**??**.

Theorem 1.10 (Abel Formula:) Let $A(t)$ be continuous on I and $\phi \in M_n(K)$ be such that $\phi'(t) = A(t)\phi(t)$ on I. Then $\det\phi$ satisfies on I the differential equation $(\det\phi)' = (trA)(\det\phi)$, or in integral form for $t, \tau \in I$,

$$\det\phi(t) = \det\phi(\tau)\exp\Big(\int_{\tau}^{t} trA(s)ds\Big) \tag{1.11}$$

Proof.: Writing the differential equation $\phi'(t) = A(t)\phi(t)$ in terms of the elements φ_{ij} and a_{ij} of respectively ϕ and A,

$$\varphi'_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t)\varphi_{kj}(t), \qquad i, j = 1, 2, 3, \cdots n. \tag{1.12}$$

Writing $\det\phi = \begin{vmatrix} \varphi_{11}(t) & \varphi_{12}(t) & \cdots & \varphi_{1n}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \cdots & \varphi_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix}$. Then we see that

$$\begin{aligned} (\det\phi)' &= \begin{vmatrix} \varphi'_{11}(t) & \varphi'_{12}(t) & \cdots & \varphi'_{1n}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \cdots & \varphi_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix} + \begin{vmatrix} \varphi_{11}(t) & \varphi_{12}(t) & \cdots & \varphi_{1n}(t) \\ \varphi'_{21}(t) & \varphi'_{22}(t) & \cdots & \varphi'_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix} \\ &+ \cdots + \begin{vmatrix} \varphi_{11}(t) & \varphi_{12}(t) & \cdots & \varphi_{1n}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \cdots & \varphi_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi'_{n1}(t) & \varphi'_{n2}(t) & \cdots & \varphi'_{nn}(t) \end{vmatrix}. \end{aligned}$$

Indeed, write $\det\phi(t) = \Gamma\Big(r_1, r_2, \cdots, r_n\Big)$, where r_i is the $i-th$ row in $\phi(t)$. Γ is then a linear function of each of its arguments, if all other rows are constant which implies that

$$\frac{d}{dt}\det\phi(t) = \Gamma\Big(\frac{d}{dt}r_1, r_2, \cdots, r_n\Big) + \Gamma\Big(r_1, \frac{d}{dt}r_2, \cdots, r_n\Big) + \cdots + \Gamma\Big(r_1, r_2, \cdots, \frac{d}{dt}r_n\Big)$$

Using (1.12) on the first of the n determinants in $(\det\phi)'$ gives

$$\begin{vmatrix} \sum\limits_k a_{1k}\varphi_{k1}(t) & \sum\limits_k a_{1k}\varphi_{k2}(t) & \cdots & \sum\limits_k a_{1k}\varphi_{kn}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \cdots & \varphi_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix}$$

. Adding $-a_{12}$ times the second row, $-a_{13}$ times the first row, etc., $-a_{1n}$ times the $n-th$ row, to the first row, does not change the determinant and thus

$$\begin{vmatrix} \sum_k a_{1k}\varphi_{k1}(t) & \sum_k a_{1k}\varphi_{k2}(t) & \cdots & \sum_k a_{1k}\varphi_{kn}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \cdots & \varphi_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix} = \begin{vmatrix} a_{11}\varphi_{11}(t) & a_{11}\varphi_{12}(t) & \cdots & a_{11}\varphi_{1n}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \cdots & \varphi_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix} = a_{11} \det \phi.$$

Repeating this for each of the terms in $(\det \phi)'$, we obtain $(\det \phi)' = (a_{11} + a_{22} + \cdots + a_{nn}) \det \phi$, giving finally $(\det \phi)' = (trA)(\det \phi)$. Integrating we get $\det \phi(t) = \det \phi(\tau) \exp\left(\int_\tau^t trA(s)ds \right)$

Theorem 1.11 (Wronskians of Solutions:) Let $\mathbf{x}_1(t) = [x_{11}(t), x_{12}(t), \cdots, x_{1n}(t)]^T$, $\mathbf{x}_2(t) = [x_{21}(t), x_{22}(t), \cdots, x_{2n}(t)]^T \cdots \mathbf{x}_n(t) = [x_{n1}(t), x_{n2}(t), \cdots, x_{nn}(t)]^T$ be n solutions of the homogeneous linear equation $\dot{\mathbf{x}} = \mathbf{P(t)x}$ on an interval I. Suppose also that $\mathbf{p(t)}$ is continuous on I. Let $W(t) = W(\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}) = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{bmatrix}$. Then :

(i) If $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}$ are linearly dependent on I, then $W = 0$ at every point of I.
(ii) If $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}$ are linearly independent on I, then $W \neq 0$ at every point of I.
Thus there are only two possibilities for solutions of homogeneous systems : Either $W = 0$ at every point of I or $W = 0$ at no point of I.

Example 1.5 Verify that the set of solution $\mathbf{x_1(t)} = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}$, $\mathbf{x_2(t)} = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}$ and $\mathbf{x_3(t)} = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$ are independent solution of the equation $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \mathbf{x}$

Solution: The Wronskian of these solution is $W(t) = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix} = -16e^{9t}$ which never zero. Hence by Theorem-1.11, we have the set of solution is linearly independent on any interval.

Theorem 1.12 The system of n linear simultaneous linear ordinary differential equations is the form of

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{1.13}$$

$$\text{Subject to the } \mathbf{x}(t_0) = \mathbf{x}_0. \tag{1.14}$$

Suppose the coefficients $\mathbf{A}(t) = [a_{ij}(t)]_{n\times n}$, $(i, j = 1, 2, \cdots, n)$ and the functions $\mathbf{f}(t) = [f_1(t), f_2(t), \cdots, f_n(t)]^T$, $(i = 1, 2, \cdots, n)$ are continuous on the interval $[t - a,\ t + a]$ containing t_0. Then the problem (1.13) with (1.14) has a unique solution $(x_1(t), x_2(t), \cdots, x_n(t))$ in $[t - a,\ t + a]$ containing t_0.

Proof. The system of n linear ordinary differential equations $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ always satisfied the Lipschitz condition (Please see the Example 1.3) on $[t - a,\ t + a]$ containing t_0. Then the proof is entirely same as the Theorem 1.2, if we put $\mathbf{f}(t, \mathbf{x}) = \mathbf{A}(t)\mathbf{x}(t)$ on $[t - a,\ t + a]$ containing

t_0.

Theorem 1.13 The solution of the homogenous system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by $\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0$ where $\Phi(t)$ is any fundamental matrix of the system.

Proof. The solution must be of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{a} \tag{1.15}$$

where $\mathbf{a}$ is a constant vector. The initial conditions gives $\mathbf{x}_0 = \Phi(t_0)\mathbf{a}$. The columns of $\Phi(t_0)$ are linearly independent by Theorem 1.9, so $\Phi(t_0)$ has an inverse $\Phi^{-1}(t_0)$. Therefore $\mathbf{a} = \Phi^{-1}(t_0)x_0$. then from equation (1.15), we have $x(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0$

Theorem 1.14 Let $\mathbf{x}_p(t)$ be any one particular solution of $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$. Then every solution of this equation is of the form $\mathbf{x}(t) = \mathbf{x}_p(t) + \Phi_c(t)$, where $\Phi_c(t)$ is a complementary function i.e $\phi_c(t)$ is the solution of $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ and conversely.

Note: If $\mathbf{A}(t) = \mathbf{A}$, i.e $a_{ij}(t) = [a_{ij}]$, $(i, j = 1, 2, \cdots, n)$ then all above theorems are also valid.

1.6 The Method of Operator

Let us consider the simultaneous linear differential equation with constant coefficients of type-I for two variables be

$$\Phi_1(D)x + \Phi_2(D)y = f(t) \tag{1.16}$$

$$\psi_1(D)x + \psi_2(D)y = g(t) \tag{1.17}$$

where x, y are the dependent variable dependent on t (independent variable) and $\Phi_1(D), \Phi_2(D)$, $\psi_1(D)$ and $\psi_2(D)$ are all rational function of $D \equiv \frac{d}{dt}$ with constant coefficients and f and g are functions of t. We have to find out the value of x and y in terms of t. Given below we have discussed operator method to solve (1.16) and (1.17).
In this section, the method of D-operator is employed to obtain the complementary and particular solutions of systems of linear ordinary differential equations of type-I. To eliminate y, we operate both side of (1.16) with $\psi_2(D)$ and both side of (1.17) with $\Phi_2(D)$. Then (1.16) and (1.17) transforming to

$$\psi_2(D)\Phi_1(D)x + \psi_2(D)\Phi_2(D)y = \psi_2(D)f(t) \tag{1.18}$$

$$\Phi_2(D)\psi_1(D)x + \Phi_2(D)\psi_2(D)y = \Phi_2(D)g(t) \tag{1.19}$$

Subtracting (1.19) from (1.18), we get

$$\Big\{\psi_2(D)\Phi_1(D) - \Phi_2(D)\psi_1(D)\Big\}x = \psi_2(D)f(t) - \Phi_2(D)g(t) \tag{1.20}$$

which is a linear equation in x and can be to find x as a function of t. Value of y can be obtained as a function of t by substituting the value of x in (1.16) or (1.17). Note that the number of arbitrary constants in the complete solution of (1.16) will be equal to order of the differential equation obtained in (1.20).

Example 1.6 Solve $\frac{dx}{dt} - 7x + y = 0$, $\frac{dy}{dt} - 2x - 5y = 0$.

Solution: Writing D for $\frac{d}{dt}$, the equations are

$$(D-7)x + y = 0 \tag{1.21}$$

$$\text{and} \qquad (D-5)y - 2x = 0 \tag{1.22}$$

Putting the value of $y = -(D-7)x$ in (1.22), we get

$$\begin{aligned} & -(D-5)(D-7)x - 2x = 0 \\ \Rightarrow \quad & -(D^2 - 12D + 35)x - 2x = 0 \\ \Rightarrow \quad & (D^2 - 12D + 35 + 2)x = 0 \\ \Rightarrow \quad & (D^2 - 12D + 37)x = 0 \end{aligned} \tag{1.23}$$

Let $x = e^{mt}$ (m being a constant) be the trial solution of the equation (1.23). The auxiliary equation of the differential equation (1.23) is

$$\begin{aligned} & m^2 - 12m + 37 = 0 \\ \Rightarrow \quad & m = 6 \pm i \end{aligned}$$

The general solution of the equation (1.23) is

$$x = e^{6t}(A\cos t + B\sin t), \text{ where } A \text{ and } B \text{ are arbitrary constants.}$$

Putting these value of x in (1.21), we get

$$\begin{aligned} y &= -(D-7)x \\ &= -(D-7)(e^{6t}(A\cos t + B\sin t)) \\ &= -6e^{6t}(A\cos t + B\sin t) - e^{6t}(-A\sin t + B\cos t) + 7e^{6t}(A\cos t + B\sin t) \\ &= e^{6t}\Big[(A-B)\cos t + (A+B)\sin t\Big]. \end{aligned}$$

Therefore the solution of the given simultaneous linear equation is given by

$$\begin{aligned} & y = e^{6t}\Big[(A-B)\cos t + (A+B)\sin t\Big]. \\ \text{and} \qquad & x = e^{6t}(A\cos t + B\sin t), \end{aligned}$$

where A and B are arbitrary constants.

Example 1.7 Solve $\frac{dx}{dt} + 4x + 3y = t$, $\frac{dy}{dt} + 2x + 5y = e^t$.

Solution: The given equations are

$$\frac{dx}{dt} + 4x + 3y = t \tag{1.24}$$

$$\frac{dy}{dt} + 2x + 5y = e^t \tag{1.25}$$

Putting the value of $y = \frac{1}{3}(t - \frac{dx}{dt} - 4x)$ from (1.24) in (1.25), we get

$$\begin{aligned}
& \frac{1}{3}\frac{d}{dt}\left(t - \frac{dx}{dt} - 4x\right) + 2x + \frac{5}{3}(t - \frac{dx}{dt} - 4x) = e^t \\
\Rightarrow\ & \frac{1}{3}(1 - \frac{d^2x}{dt^2} - 4\frac{dx}{dt}) + 2x + \frac{5}{3}(t - \frac{dx}{dt} - 4x) = e^t \\
\Rightarrow\ & \frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 14x = 1 + 5t - 3e^t \\
\Rightarrow\ & (D^2 + 9D + 14)x = 1 + 5t - 3e^t
\end{aligned} \tag{1.26}$$

Let $y(x) = e^{mt}$ (m being a constant) be a trial solution of the corresponding homogenous differential equation of (1.26). Then its auxiliary equation is

$$\begin{aligned}
& m^2 + 9m + 14 = 0 \\
\Rightarrow\ & m = -7, -2
\end{aligned}$$

The complementary function(C.F) of the equation (1.26) is

$$C.F = Ae^{-7t} + Be^{-2t}, \text{ where } A \text{ and } B \text{ are arbitrary constants.}$$

The particular integral of (1.26) is

$$\begin{aligned}
P.I &= \frac{1}{D^2 + 9D + 14}(1 + 5t - 3e^t) \\
&= \frac{1}{D^2 + 9D + 14} + 5\frac{1}{D^2 + 9D + 14}t - 3\frac{1}{D^2 + 9D + 14}e^t \\
&= \frac{1}{14} + \frac{5}{14}(1 + \frac{9D + D^2}{14})^{-1}t - \frac{3}{1^2 + 9 \cdot 1 + 14}e^t \\
&= \frac{1}{14} + \frac{5}{14}(1 - \frac{9D}{14} - \cdots)t - \frac{3}{24}e^t \\
&= \frac{5t}{14} - \frac{e^t}{8} - \frac{31}{196}.
\end{aligned}$$

Therefore the general solution of the equation (1.26) is

$$x(t) = Ae^{-7t} + Be^{-2t} + \frac{5t}{14} - \frac{e^t}{8} - \frac{31}{196}.$$

Putting the said value of x in (1.24), we get

$$\begin{aligned}
y &= \frac{1}{3}(t - \frac{dx}{dt} - 4x) \\
&= \frac{1}{3}\left(t - \frac{d}{dt}\left(Ae^{-7t} + Be^{-2t} + \frac{5t}{14} - \frac{e^t}{8} - \frac{31}{196}\right) - 4\left(Ae^{-7t} + Be^{-2t} + \frac{5t}{14} - \frac{e^t}{8} - \frac{31}{196}\right)\right) \\
&= Ae^{-7t} - \frac{2}{3}Be^{-2t} + \frac{5e^t}{24} - \frac{t}{7} + \frac{9}{98}
\end{aligned}$$

Example 1.8 Solve $\frac{dx}{dt} + y = e^t, \frac{dy}{dt} - x = e^{-t}$

Solution: Writing D for $\frac{d}{dt}$, the equations are

$$Dx + y = e^t \tag{1.27}$$
$$\text{and} \qquad Dy - x = e^{-t} \tag{1.28}$$

Differentiating (1.27) both sides with respect to t, we get

$$\begin{aligned} & D^2x + Dy = e^t \\ \Rightarrow \quad & D^2x + (x + e^{-t}) = e^t \text{ form (1.28).} \\ \Rightarrow \quad & D^2x + x = e^t - e^{-t} \end{aligned} \tag{1.29}$$

Let $y(x) = e^{mt}$ (m being a constant) be a trial solution of the corresponding homogenous differential equation of (1.29). Then its auxiliary equation is

$$\begin{aligned} & m^2 + 1 = 0 \\ \Rightarrow \quad & m = \pm i \end{aligned}$$

The complementary function of the equation (1.29) is

$$C.F = A\cos t + B\sin t, \text{ where } A \text{ and } B \text{ are arbitrary constants.}$$

The particular integral of (1.29) is

$$\begin{aligned} P.I &= \frac{1}{D^2+1}(e^t - e^{-t}) \\ &= \frac{1}{D^2+1}e^t - \frac{1}{D^2+1}e^{-t} \\ &= \frac{e^t}{2} - \frac{e^{-t}}{2}. \end{aligned}$$

Therefore the general solution of the equation (1.29) is

$$x = A\cos t + B\sin t + \frac{e^t}{2} - \frac{e^{-t}}{2}$$

Putting the said value of x in (1.27), we get

$$\begin{aligned} y &= e^t - D\Big(A\cos t + B\sin t + \frac{e^t}{2} - \frac{e^{-t}}{2}\Big) \\ &= e^t - \frac{d}{dt}\Big(A\cos t + B\sin t + \frac{e^t}{2} - \frac{e^{-t}}{2}\Big) \\ &= e^t - \Big(-A\sin t + B\cos t + \frac{e^t}{2} + \frac{e^{-t}}{2}\Big) \\ &= A\sin t - B\cos t + \frac{e^t}{2} - \frac{e^{-t}}{2}. \end{aligned}$$

Therefore the solution of the given simultaneous linear equation is given by

$$\begin{aligned} & x = A\cos t + B\sin t + \frac{e^t}{2} - \frac{e^{-t}}{2} \\ \text{and} \qquad & y = A\sin t - B\cos t + \frac{e^t}{2} - \frac{e^{-t}}{2}. \end{aligned}$$

Example 1.9 Solve: $t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + 2y = 0,\ t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - 2x = 0,\ t > 0$

Solution : Let $t = e^z$ so that $z = \log t$ and let $\theta \equiv \frac{d}{dz} \equiv t\frac{d}{dt}$. Then $t^2\frac{d^2}{dt^2} \equiv \theta(\theta - 1)$. Then the given equations become

$$\Big(\theta(\theta-1)+\theta\Big)x + 2y = 0, \ \Rightarrow \theta^2 x + 2y = 0 \tag{1.30}$$

$$\Big(\theta(\theta-1)+\theta\Big)y - 2x = 0, \ \Rightarrow \theta^2 y - 2x = 0 \tag{1.31}$$

Eliminating y from (1.30) and (1.31), we get

$$(\theta^4 + 4)x = 0. \tag{1.32}$$

The general solution of (1.32) is

$$x(t) = e^z(c_1\cos z + c_2\sin z) + e^{-z}(c_3\cos z + c_4\sin z) \tag{1.33}$$

where c_1, c_2, c_3, c_4 are arbitrary constants.
Now, $\theta^2 x = 2e^z(c_2\cos z - c_1\sin z) + 2e^{-z}(c_3\sin z - c_4\cos z)$. Using this value, we have

$$y = e^z(c_1\sin z - c_2\cos z) + e^{-z}(c_4\cos z - c_3\sin z) \tag{1.34}$$

By replacing $x = e^t$ in (1.33) and (1.34), we get the required solution as

$$x(t) = c_1 t\cos(\log t) + c_2 t\sin(\log t) + c_3 t^{-1}\cos(\log t) + c_4 t^{-1}\sin(\log t),\ t > 0$$
$$y(t) = c_1 t\sin(\log t) - c_2 t\cos(\log t) + c_4 t^{-1}\cos(\log t) - c_3 t^{-1}\sin(\log t),\ t > 0$$

Example 1.10 The equation of motion of a particle are given by $\frac{dx}{dt} + wy = 0$, $\frac{dy}{dt} - wx = 0$. Find the path of the particle and show that it is a circle. **Gate(MA): 2017**; **VU(CBCS): 2018**

Solution: Writing D for $\frac{d}{dt}$, the equations are

$$Dx + wy = 0 \tag{1.35}$$

$$\text{and} \qquad -wx + Dy = 0 \tag{1.36}$$

Differentiating (1.35) both sides with respect to t, we get

$$\begin{aligned} & D^2x + wDy = 0 \\ \Rightarrow\ & D^2x + w(wx) = 0, \quad \text{(form (1.36))}. \\ \Rightarrow\ & D^2x + w^2x = 0 \\ \Rightarrow\ & x = A\cos wt + B\sin wt. \end{aligned} \tag{1.37}$$

Putting these value of x in (1.35), we get

$$\begin{aligned} & wy = -Dx = \frac{d}{dt}(A\cos wt + B\sin wt) \\ \Rightarrow\ & wy = -Aw\sin wt + Bw\cos wt \\ \Rightarrow\ & y = B\cos wt - A\sin wt \end{aligned} \tag{1.38}$$

Squaring (1.37) and (1.38) and then adding, we get

$$\begin{aligned} & x^2 + y^2 = A^2 + B^2 \\ \Rightarrow \quad & x^2 + y^2 = R^2, [\text{where } R^2 = A^2 + B^2]. \end{aligned}$$

which is a circle.
In general, the system of n linear ordinary differential equations is the form of

$$\begin{aligned} \phi_{11}(D)x_1 + \phi_{12}(D)x_2 + \cdots + \phi_{1n}(D)x_n &= f_1(t) \\ \phi_{21}(D)x_1 + \phi_{22}(D)x_2 + \cdots + \phi_{2n}(D)x_n &= f_2(t) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad & \\ \phi_{n1}(D)x_1 + \phi_{n2}(D)x_2 + \cdots + \phi_{nn}(D)x_n &= f_n(t) \end{aligned} \tag{1.39}$$

where $x_1, x_2, \cdots, x_n$ are n dependent variables dependent on t (independent variable) and $\phi_{ij}(D)$, $i, j = 1, 2, \cdots, n$ are all rational function of $D \equiv \frac{d}{dt}$ with constant coefficients and $f_i(t)$, $i = 1, 2, \cdots, n$.

Complementary Solutions

Let the complementary solutions of the system of linear differential equation (1.39) be $x_{1c}, x_{2c}, \cdots, x_{nc}$. Then $x_{1c}, x_{2c}, \cdots, x_{nc}$ are satisfied the homogenous differential equations of (1.39) *i.e.*

$$\phi(D)x_{1c} = 0,\ \phi(D)x_{2c} = 0,\ \cdots, \phi(D)x_{nc} = 0,$$

where $\phi(D)$ is determinant of the coefficient matrix of the system of linear differential equation (1.39) i.e,

$$\phi(D) = \begin{vmatrix} \phi_{11}(D) & \phi_{12}(D) & \cdots & \phi_{1n}(D) \\ \phi_{21}(D) & \phi_{22}(D) & \cdots & \phi_{2n}(D) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1}(D) & \phi_{n2}(D) & \cdots & \phi_{nn}(D) \end{vmatrix}$$

Hence, the unknowns $x_{1c}, x_{2c}, \cdots, x_{nc}$ all have the same characteristic equation $\phi(\lambda) = 0$ and, as a result, the same form of complementary solutions.
The complementary solutions of system (1.39) contain arbitrary constants, the number of which is the degree of polynomial of $\phi(D)$. It is likely that the complementary solutions $x_{1c}, x_{2c}, \cdots, x_{nc}$, written using the roots of the characteristic equation $\phi(\lambda) = 0$, will contain more constants. The extra constants can be eliminated by substituting the solutions into any one of the original equations in system (1.39).

Particular Solutions

A particular solution of the system of linear differential equation (1.39) is given by using Cramers Rule,

$$x_{ip}(t) = \frac{\Delta_i(t)}{\phi(D)},\ i = 1, 2, \cdots, n,$$

where $\phi(D)$ is determinant of the coefficient matrix as studied in the previous section for complementary solution, $\Delta_i(t)$ is $\phi(D)$ with the i^{th} column being replaced by the right-hand side vector of functions, i.e.,

$$\Delta_i(t) = \begin{vmatrix} \phi_{11}(D) & \cdots & \phi_{1,i-1}(D) & f_1(t) & \phi_{1,i+1}(D) & \cdots & \phi_{1n}(D) \\ \phi_{21}(D) & \cdots & \phi_{2,i-1}(D) & f_2(t) & \phi_{2,i+1}(D) & \cdots & \phi_{2n}(D) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{n1}(D) & \cdots & \phi_{n,i-1}(D) & f_n(t) & \phi_{n,i+1}(D) & \cdots & \phi_{nn}(D) \end{vmatrix}$$

Note: It should be state that, since the elements of the determinant are operators and functions, operators must precede functions when evaluating determinants. Furthermore, since $\Delta_i(t), i = 1, 2, \cdots, n$ are functions, when determining x_{ip}, $\phi^{-1}(D)$ should precede $\Delta_i(t)$.

Example 1.11 Solve $\frac{dx}{dt} + 4x + 3y = t$, $\frac{dy}{dt} + 2x + 5y = e^t$

Solution: Writing D for $\frac{d}{dt}$, the equations are

$$(D + 4)x + 3y = t \tag{1.40}$$

and
$$2x + (D + 5)y = e^t \tag{1.41}$$

Let the complementary solutions of the given system of linear differential equation (1.41) be $x_c(t), y_c(t)$. Then x_c, y_c are satisfied the homogenous differential equations of (1.40) and (1.41) *i.e.*

$$\phi(D)x_c(t) = 0, \ \phi(D)y_c(t) = 0$$

where $\phi(D)$ is determinant of the coefficient matrix of the given system of linear differential equations (1.40) -(1.41) i.e,

$$\phi(D) = \begin{vmatrix} D+4 & 3 \\ 2 & D+5 \end{vmatrix}$$

Hence the unknowns $x_c(t), y_c(t)$, all have the same characteristic equation $\phi(\lambda) = 0$ and as a result, the same form of complementary solutions. Now,

$$\phi(\lambda) = \begin{vmatrix} \lambda+4 & 3 \\ 2 & \lambda+5 \end{vmatrix} = \lambda^2 + 9\lambda + 14$$

So, $\phi(\lambda) = 0, \Rightarrow \lambda = -7, -2$. Hence, they have the same complementary solutions given by $x_c = Ae^{-7t} + Be^{-2t}$ and $y_c = Ce^{-7t} + De^{-2t}$.
To find the particular solution of the given system of linear differential equation (1.41), we have,

$$\Delta_x(t) = \begin{vmatrix} t & 3 \\ e^t & D+5 \end{vmatrix} = 5t + 1 - 3e^t,$$

$$\Delta_y(t) = \begin{vmatrix} D+4 & t \\ 2 & e^t \end{vmatrix} = 5e^t - 2t,$$

$$x_p(t) = \frac{\Delta_x(t)}{\phi(D)} = \frac{5t + 1 - 3e^t}{D^2 + 9D + 14} = \frac{1}{14}(1 + \frac{9D + D^2}{14})^{-1}(5t + 1) - \frac{3e^t}{24} = \frac{5t}{14} - \frac{31}{196} - \frac{e^t}{8},$$

$$y_p(t) = \frac{\Delta_y(t)}{\phi(D)} = \frac{5e^t - 2t}{D^2 + 9D + 14} = \frac{5e^t}{24} - \frac{1}{14}(1 + \frac{9D + D^2}{14})^{-1}(-2t) = \frac{5e^t}{24} - \frac{t}{7} + \frac{9}{98}.$$

The general solutions are

$$x(t) = x_c(t) + x_p(t) = Ae^{-7t} + Be^{-2t} + \frac{5t}{14} - \frac{31}{196} - \frac{e^t}{8} \quad (1.42)$$
$$y(t) = y_c(t) + y_p(t) = Ce^{-7t} + De^{-2t} + \frac{5e^t}{24} - \frac{t}{7} + \frac{9}{98} \quad (1.43)$$

Since $\phi(D) = 0$ is a polynomial of degree 2 in D, the general solutions should contain only two arbitrary constants. The two extra constants C, D can be eliminated by substituting the complementary solutions $x_c(t)$, $y_c(t)$ into either the homogeneous equation of (1.40) or (1.41). So, substituting the complementary solutions $x_c(t)$, $y_c(t)$ in $(D+4)x + 3y = 0$, we get

$$(D+4)(Ae^{-7t} + Be^{-2t}) + 3(Ce^{-7t} + De^{-2t}) = 0$$
$$\Rightarrow (-3A + 3C)e^{-7t} + (2B + 3D)e^{-2t} = 0$$
$$\Rightarrow C = A, \;\; D = -\frac{2}{3}B.$$

Then the general solutions become

$$x(t) = x_c(t) + x_p(t) = Ae^{-7t} + Be^{-2t} + \frac{5t}{14} - \frac{31}{196} - \frac{e^t}{8}$$
$$y(t) = y_c(t) + y_p(t) = Ae^{-7t} - \frac{2}{3}Be^{-2t} + \frac{5e^t}{24} - \frac{t}{7} + \frac{9}{98}$$

Example 1.12 Solve $\frac{dx}{dt} + 4x + 3y = \sin t$, $\frac{dy}{dt} + 2x + 5y = e^t$

Solution: Writing D for $\frac{d}{dt}$, the equations are

$$(D+4)x + 3y = \sin t \quad (1.44)$$
$$\text{and} \quad 2x + (D+5)y = e^t \quad (1.45)$$

Let the complementary solutions of the given system of linear differential equation (1.45) be $x_c(t), y_c(t)$. Then x_c, y_c are satisfied the homogenous differential equations of (1.44) and (1.45) *i.e.*

$$\phi(D)x_c(t) = 0, \; \phi(D)y_c(t) = 0$$

where $\phi(D)$ is determinant of the coefficient matrix of the given system of linear differential equations (1.44) -(1.45) i.e,

$$\phi(D) = \begin{vmatrix} D+4 & 3 \\ 2 & D+5 \end{vmatrix}$$

Hence the unknowns $x_c(t), y_c(t)$, all have the same characteristic equation $\phi(\lambda) = 0$ and as a result, the same form of complementary solutions. Now,

$$\phi(\lambda) = \begin{vmatrix} \lambda+4 & 3 \\ 2 & \lambda+5 \end{vmatrix} = \lambda^2 + 9\lambda + 14$$

So, $\phi(\lambda) = 0, \Rightarrow \lambda = -7, -2$. Hence, they have the same complementary solutions given by $x_c = Ae^{-7t} + Be^{-2t}$ and $y_c = Ce^{-7t} + De^{-2t}$.

To find the particular solution of the given system of linear differential equation (1.45), we have,

$$\Delta_x(t) = \begin{vmatrix} \sin t & 3 \\ e^t & D+5 \end{vmatrix} = \cos t + 5\sin t - 3e^t,$$

$$\Delta_y(t) = \begin{vmatrix} D+4 & \sin t \\ 2 & e^t \end{vmatrix} = 5e^t - 2\sin t,$$

$$\begin{aligned} x_p(t) &= \frac{\Delta_x(t)}{\phi(D)} = \frac{\cos t + 5\sin t - 3e^t}{D^2+9D+14} = \frac{\cos t}{9D+13} + \frac{5\sin t}{9D+13} - \frac{3e^t}{24} \\ &= \frac{(9D-13)\cos t}{81D^2-169} + \frac{(9D-13)5\sin t}{81D^2-169} - \frac{3e^t}{24} \\ &= \frac{-9\sin t - 13\cos t}{-81-169} + \frac{45\cos t - 65\sin t}{-81-169} - \frac{3e^t}{24} \\ &= \frac{74\sin t}{250} + \frac{32\cos t}{250} - \frac{3e^t}{24} \end{aligned}$$

$$y_p(t) = \frac{\Delta_y(t)}{\phi(D)} = \frac{5e^t - 2\sin t}{D^2+9D+14} = \frac{5e^t}{24} - \frac{2\sin t}{9D+13} = \frac{5e^t}{24} + \frac{18\cos t - 26\sin t}{250}.$$

The general solutions are

$$x(t) = x_c(t) + x_p(t) = Ae^{-7t} + Be^{-2t} + \frac{74\sin t}{250} + \frac{32\cos t}{250} - \frac{3e^t}{24} \tag{1.46}$$

$$y(t) = y_c(t) + y_p(t) = Ce^{-7t} + De^{-2t} + \frac{5e^t}{24} + \frac{18\cos t}{250} - \frac{26\sin t}{250} \tag{1.47}$$

Since $\phi(D) = 0$ is a polynomial of degree 2 in D, the general solutions should contain only two arbitrary constants. The two extra constants C, D can be eliminated by substituting the complementary solutions $x_c(t)$, $y_c(t)$ into either the homogeneous equation of (1.44) or (1.45). So, substituting the complementary solutions $x_c(t)$, $y_c(t)$ in $(D+4)x + 3y = 0$, we get

$$\begin{aligned} &(D+4)(Ae^{-7t} + Be^{-2t}) + 3(Ce^{-7t} + De^{-2t}) = 0 \\ &\Rightarrow (-3A + 3C)e^{-7t} + (2B + 3D)e^{-2t} = 0 \\ &\Rightarrow C = A, \;\; D = -\frac{2}{3}B. \end{aligned}$$

Then the general solutions become

$$\begin{aligned} x(t) &= x_c(t) + x_p(t) = Ae^{-7t} + Be^{-2t} + \frac{74\sin t}{250} + \frac{32\cos t}{250} - \frac{3e^t}{24} \\ y(t) &= y_c(t) + y_p(t) = Ae^{-7t} - \frac{2}{3}Be^{-2t} + \frac{5e^t}{24} + \frac{18\cos t}{250} - \frac{26\sin t}{250} \end{aligned}$$

1.7 Matrix Method(Normal Form)

The matrix method is the most general and systematic approach, especially in dealing with systems of higher dimensions simultaneous differential equations of type-I . However, the said method is the most difficult to master because of the challenging concepts in eigenvalues and eigenvectors, particularly when multiple eigenvalues are involved.

Note: The system of equations of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}(t) + f(t)$ is called **Normal form** .

1.7.1 Structure of the solutions of $n-$ dimensional homogeneous linear systems with constant coefficients

The general homogeneous, first order linear system of n dimensions is

$$\dot{x}(t) = Ax(t) \tag{1.48}$$

where A is an $n \times n$ matrix whose elements a_{ij} are real constants and $x(t)$ is a column vector of the n variables.
Seek a solution of the form $x(t) = e^{\lambda t}v$, where v is a column vector of the n variables. Substituting into equation (1.48) yields $\lambda e^{\lambda t}v = Ae^{\lambda t}v$. Since, $e^{\lambda t} \neq 0$, one obtains

$$(A - \lambda I)v = 0 \tag{1.49}$$

where I is an $n \times n$ identity matrix. Equation (1.49) is a system of homogeneous linear algebraic equations. To have nonzero solutions for v, the determinant of the coefficient matrix must be zero, i.e.,

$$\det(A - \lambda I) = 0 \tag{1.50}$$

which leads to the characteristic equation, a polynomial equation in λ of degree n.

Distinct Eigenvalues

The n solutions $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the characteristic equation (1.50) are called the eigenvalues of A. Suppose the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct real numbers. A nonzero solution v_k of system (1.49) with $\lambda = \lambda_k$, i.e.,

$$(A - \lambda I)v_k = 0, \;\; k = 1, 2, \cdots, n, \tag{1.51}$$

is called an eigenvector corresponding to eigenvalue λ_k.

Theorem 1.15 For the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A}$ a real, constant matrix whose eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all different then $\mathbf{x}(t) = \left[\mathbf{v}_1 e^{\lambda_1 t}, \, \mathbf{v}_2 e^{\lambda_2 t}, \, \cdots, \, \mathbf{v}_n e^{\lambda_n t}\right]$ is a fundamental matrix where $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and the complementary solution of the homogeneous linear system is $\mathbf{x}(t) = \left[c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}\right]$ where $c_1, c_2, \cdots, c_n$ are real constants.

Proof. As the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all distinct real numbers, then $\mathbf{v}_1 e^{\lambda_1 t}, \, \mathbf{v}_2 e^{\lambda_2 t}, \, \cdots, \, \mathbf{v}_n e^{\lambda_n t}$ are all independent solutions. By definition of (1.1), $\mathbf{X}(t) = \left[\mathbf{v}_1 e^{\lambda_1 t}, \, \mathbf{v}_2 e^{\lambda_2 t}, \, \cdots, \, \mathbf{v}_n e^{\lambda_n t}\right]$ is the fundamental matrix. It is true that the linear combination of the independent solutions is also the solution of the linear homogenous differential equation. So, the complementary solution of the homogeneous linear system is $\mathbf{x}(t) = \left[c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}\right]$ where $c_1, c_2, \cdots, c_n$ are real constants.

Example 1.13 Find the fundamental matrix and the complementary solution of the homogenous linear system of differential equations **VU(CBCS):2018**

$$\frac{dx_1}{dt} = 3x_1 + x_2 \tag{1.52}$$
$$\frac{dx_2}{dt} = x_1 + 3x_2 \tag{1.53}$$

Solution: Here

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

So the eigenvalues of the matrix A are satisfied by the equation $(3-\lambda)^2 - 1 = 0$ and hence $\lambda_1 = 2$, $\lambda_2 = 4$ are the required two eigenvalues of A.

Let $V_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be a eigenvector corresponding to the eigenvalue $\lambda_1 = 2$. Then its satisfy the equation

$$\begin{aligned}
\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\Rightarrow \quad v_1 + v_2 &= 0 \\
\Rightarrow \quad v_1 &= -v_2 \\
\Rightarrow \quad v_2 &= -v_1 = c_1 \text{ (say)}
\end{aligned}$$

Therefore, for $\lambda_1 = 2$, the corresponding eigenvector is

$$V_1 = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{for } c_1 = 1$$

and similarly for $\lambda_2 = 4$, the corresponding eigenvector is $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

As $\lambda_1 \neq \lambda_2$, so the fundamental matrix is

$$X(t) = \begin{pmatrix} V_1 e^{\lambda_1 t} & V_2 e^{\lambda_2 t} \end{pmatrix} = \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \right) = \begin{pmatrix} -e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}.$$

So the complementary solution is

$$\mathbf{X(t)} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 V_1 e^{2t} + c_2 V_2 e^{4t} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}. \tag{1.54}$$

Theorem 1.16 The solution of the homogenous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0$ where $\mathbf{X}(t)$ is the fundamental matrix of the system.

Proof. The solution must be of the form

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} \tag{1.55}$$

where $\mathbf{c} = [c_1, c_2, \cdots, c_n]^T$ is a constant vector. The initial conditions gives $\mathbf{x}_0 = \mathbf{X}(t_0)\mathbf{c}$. As $\mathbf{X}(t)$ is the fundamental matrix of the system, so the columns of $\mathbf{X}(t_0)$ are linearly independents.

So $\mathbf{X}(t_0)$ has an inverse $\mathbf{X}^{-1}(t_0)$. Therefore $\mathbf{c} = \mathbf{X}^{-1}(t_0)\mathbf{x}_0$. then from equation (1.55), we have $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0$

Example 1.14 Find the fundamental matric and the solution $\mathbf{x}(t)$ such that $\mathbf{x}(0) = [1 \;\; 6]^T$ for the system.

$$\frac{dx_1}{dt} = 2x_1 - x_2 \tag{1.56}$$

$$\frac{dx_2}{dt} = -4x_2 \tag{1.57}$$

Solution: Here

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix}$$

and the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -4$.
For $\lambda_1 = 2$, corresponding eigenvector of A is

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $\lambda_2 = -4$, corresponding eigenvector of A is

$$\mathbf{V}_2 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

As $\lambda_1 \neq \lambda_2$, so the fundamental matrix is

$$X(t) = \begin{pmatrix} V_1 e^{\lambda_1 t} & V_2 e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} & \begin{pmatrix} 1 \\ 6 \end{pmatrix} e^{-4t} \end{pmatrix} = \begin{pmatrix} e^{2t} & e^{-4t} \\ 0 & 6e^{-4t} \end{pmatrix}.$$

Also $X(0) = \begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix}$. So $X^{-1}(0) = \frac{1}{6}\begin{pmatrix} 6 & -1 \\ 0 & 1 \end{pmatrix}$.
Hence $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)x_0 = \begin{pmatrix} e^{2t} & e^{-4t} \\ 0 & 6e^{-4t} \end{pmatrix} \frac{1}{6}\begin{pmatrix} 6 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} e^{-4t} \\ 6e^{-4t} \end{pmatrix}$

Complex Eigenvalues

Suppose that matrix A of the homogeneous system $\dot{x}(t) = Ax(t)$ is a real matrix. If $\alpha + i\beta$ is an eigenvalue with the corresponding eigenvector v, then corresponding to the eigenvalues $\alpha \pm i\beta$,

$$x_1(t) = Re(e^{\lambda t}v) = e^{\alpha t}\Big[Re(v)\cos\beta t - Im(v)\sin\beta t\Big]$$

$$x_2(t) = Im(e^{\lambda t}v) = e^{\alpha t}\Big[Re(v)\sin\beta t + Im(v)\cos\beta t\Big]$$

are two linearly independent real-valued solutions, or

$$x(t) = A\,Re(e^{\lambda t}v) + B\,Im(e^{\lambda t}v)$$

Example 1.15 Find the general solution for the system

$$\frac{dy_1}{dt} = -y_1 + 5y_2$$

$$\frac{dy_2}{dt} = -4y_1 - 5y_2$$

Solution: This equation is $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{pmatrix} -1 & 5 \\ -4 & -5 \end{pmatrix}$$

We can find the eigenvalues, the characteristic equation is

$$\begin{vmatrix} -1-\lambda & -5 \\ -4 & -5-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 25 = 0$$

so that $\lambda = -3 \pm 4i$. Next, we need the eigenvector for $\lambda = -3 + 4i$:

$$\begin{pmatrix} 2-4i & 5 \\ -4 & -2-4i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (2-4i)v_1 + 5v_2 \\ -4v_1 - (2+4i)v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To solve the above equations, let $v_1 = \alpha_1 + i\beta_1$ and $v_2 = \alpha_2 + i\beta_2$. Then equation real and imaginary parts, we get,

$$\begin{aligned} &2\alpha_1 + 5\alpha_2 + 4\beta_1 = 0 \\ &-4\alpha_1 + 2\beta_1 + 5\beta_2 = 0 \\ &-4\alpha_1 - 2\alpha_2 + 4\beta_2 = 0 \\ &2\alpha_2 + 2\beta_1 + \beta_2 = 0 \end{aligned}$$

Solving the above equations, we get $\alpha_1 = 5$, $\alpha_2 = 0$, $\beta_1 = -2$ and $\beta_2 = 4$. So,

$$\mathbf{v} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Hence, $e^{\lambda t}v = e^{-3t}(\cos 4t + i\sin 4t)\Big\{\begin{pmatrix} 5 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 4 \end{pmatrix}\Big\}$.
Hence, the complementary solution is

$$\begin{aligned} y(t) &= A\,Re(e^{\lambda t}v) + B\,Im(e^{\lambda t}v) \\ &= Ae^{-3t}\Big(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 4 \end{pmatrix}\sin 4t\Big) + Be^{-3t}\Big(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\sin 4t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}\cos 4t\Big) \end{aligned}$$

Therefore,

$$\begin{aligned} y_1(t) &= 5e^{-3t}(A\cos 4t + B\sin 4t) \\ y_2(t) &= 2e^{-3t}\Big[(-A + 2B)\cos 4t - (2A + B)\sin 4t\Big] \end{aligned}$$

Multiple Eigenvalues

Let us call an eigenvalue λ of a matrix A with algebraic multiplicity m complete if it has m linearly independent associated eigenvectors. An eigenvalue λ of a matrix A with algebraic multiplicity k is called defective if it is not complete. If λ has only k linearly independent eigenvectors ($k < m$), then the number $d = m - k$ is called the defect of defective eigenvalue λ.

Suppose matrix A of the homogeneous system $\dot{x}(t) = Ax(t)$ has an eigenvalue λ of algebraic multiplicity $m > 1$ and a sequence of generalized eigenvectors corresponding to λ is $v_1, v_2, \cdots, v_m$.

Then, corresponding to the eigenvalues $\lambda, \lambda, \cdots, \lambda$(repeated m times), m linearly independent solutions of the homogeneous system are

$$\begin{aligned}
&x_i(t) = e^{\lambda t} v_i, \; i = 1, 2, \cdots, k; \; 1 \le k < m, \\
&x_{k+1}(t) = e^{\lambda t}(v_k t + v_{k+1}), \\
&x_{k+2}(t) = e^{\lambda t}(v_k \frac{t^2}{2!} + v_{k+1} t + v_{k+2}), \\
&\vdots \qquad \vdots \qquad \vdots \\
&x_m(t) = e^{\lambda t}\Big[v_k \frac{t^{m-k}}{(m-k)!} + v_{k+1} \frac{t^{m-k-1}}{(m-k-1)!} + v_{k+2} \frac{t^{m-k-2}}{(m-k-2)!} + \cdots + v_{m-2} \frac{t^2}{2!} + v_{m-1} t + v_m\Big].
\end{aligned}$$

Example 1.16 For the above multiple eigenvalues problem, show that $(A - \lambda I)x_{r+1}(t) = x_r(t), \; r = k, k+1, \cdots, m-1$. Hence deduce that $(A - \lambda I)v_{k+1} = v_k$.

Solution: We have

$$\begin{aligned}
&x_{r+1}(t) = e^{\lambda t}\Big[v_k \frac{t^{r+1-k}}{(r+1-k)!} + v_{k+1} \frac{t^{r-k}}{(r-k)!} + v_{k+2} \frac{t^{r-k-1}}{(r-k-1)!} + \cdots + v_{r-1} \frac{t^2}{2!} + v_r t + v_{r+1}\Big] \\
\text{So,} \quad &\dot{x}_{r+1}(t) = \lambda x_{r+1}(t) + e^{\lambda t}\Big[v_k \frac{t^{r-k}}{(r-k)!} + v_{k+1} \frac{t^{r-k-1}}{(r-k-1)!} + v_{k+2} \frac{t^{r-k-2}}{(r-k-2)!} + \cdots + v_{r-1} t + v_r\Big] \\
&A x_{r+1}(t) = \lambda x_{r+1}(t) + x_r(t), \; r = k, k+1, \cdots, m-1 \\
\therefore \quad &(A - \lambda I)x_{r+1}(t) = x_r(t), \; r = k, k+1, \cdots, m-1.
\end{aligned}$$

Also putting $r = k$, we get, $(A - \lambda I)x_{k+1}(t) = x_k(t) \Rightarrow (A - \lambda I)e^{\lambda t}(v_k t + v_{k+1}) = e^{\lambda t} v_k \Rightarrow (A - \lambda I)(v_k t + v_{k+1}) = v_k \Rightarrow (A - \lambda I)v_{k+1} = v_k, \; (\because A v_k = \lambda v_k)$.

Example 1.17 Find the general solution for the system

$$\begin{aligned}
\frac{dy_1}{dt} &= 3y_1 + y_2 \\
\frac{dy_2}{dt} &= -y_1 + y_2
\end{aligned}$$

Solution: This equation is $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

We can find the eigenvalues, the characteristic equation is

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = 0$$

so that $\lambda = 2$, 2 i.e repeated eigenvalues. Next, we need the eigenvector for $\lambda = 2$:

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}$$

so $3a + b = 2a$ or $b = -a$, hence, choosing $a = 1$ we get

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

As the eigenvalue $\lambda = 2$ is repeated two times, so the general solution can be written as

$$\mathbf{y} = c_1\mathbf{v_1}e^{2t} + c_2\left(t\mathbf{v_1} + \mathbf{v_2}\right)e^{2t}$$

where we are to find the value of $\mathbf{v_2}$ which satisfies the equation

$$\begin{pmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix}\mathbf{v_2} = \mathbf{v_1} \Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ (See example 1.16).}$$

$$\text{Writing } \mathbf{v_2} = \begin{pmatrix} e \\ f \end{pmatrix},$$

we have

$$\begin{aligned} e + f &= 1 \\ -e - f &= -1 \end{aligned}$$

These two equations are the same, as you expect, and if $f = 0$ then $e = 1$. Thus, the general solution is given by

$$\mathbf{y} = c_1\mathbf{v_1}e^{\lambda t} + c_2(\mathbf{v_1}t + \mathbf{v_2})e^{\lambda t} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]e^{2t}$$

$$\text{or,} \qquad \mathbf{y} = \left[(c_1 + c_2t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]e^{2t}.$$

where c_1 and c_2 are integrating constants.

Example 1.18 Find the general solution for the system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}\mathbf{x}$$

Solution: This equation is $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{pmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{pmatrix}$$

We can find the eigenvalues, the characteristic equation is

$$\begin{vmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{vmatrix} = (3-\lambda)((9-\lambda)(-1-\lambda)+24) = (3-\lambda)(15-8\lambda+\lambda^2) = (5-\lambda)(3-\lambda)^2 = 0.$$

Thus $\mathbf{A}$ has the distinct eigenvalue $\lambda_1 = 5$ and repeated eigenvalue $\lambda_2 = 3$ of multiplicity $m = 2$.
Case 1: $\lambda_1 = 5$. The eigenvector equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$, where $\mathbf{v} = [a, b, c]^T$ is

$$(\mathbf{A} - 5\mathbf{I})\mathbf{v} = \begin{bmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Each of the first two equations $4a + 4b = 0$ and $-6a - 6b = 0$ yields $b = -a$. Then the third equation reduces to $2a - 2c = 0$, so that $c = a$. The choice $a = 1$ then yields the eigenvector

$\mathbf{v_1} = [1, -1, 1]^T$ associated with the eigenvalue $\lambda_1 = 5$.
Case 2: $\lambda_2 = 3$. The eigenvector equation is

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \begin{bmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so the nonzero vector $\mathbf{v} = [a, b, c]^T$ is an eigenvector if and only if $6a + 4b = 0$. If $c = 1$, then $a = b = 0$, this gives the eigenvector $\mathbf{v_2} = [0, 0, 1]^T$ associated with $\lambda_2 = 3$. If $c = 0$, then we must choose a to be nonzero. For instance, if $a = 2$, we get b=-3, so $\mathbf{v_3} = [2, -3, 0]^T$ is second linearly independent eigenvector associated with the multiplicity 2 eigenvalue $\lambda_2 = 3$.
Thus we have found a complete set $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ of three eigenvectors associated with the eigenvalues $5, 3, 3$. The corresponding general solution of (1.18) is
$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{5t} + c_2\mathbf{v}_2 e^{3t} + c_3\mathbf{v}_3 e^{3t} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} e^{3t}$ with scalar component functions given by $x_1(t) = c_1 e^{5t} + 2c_3 e^{3t}, \quad x_2(t) = -c_1 e^{5t} - 3c_3 e^{3t}, \quad x_3(t) = c_1 e^{5t} + c_2 e^{3t}$.

1.7.2 Structure of the solutions of $n-$ dimensional homogeneous linear systems

The general homogeneous, first order linear system of n dimensions is $\dot{x} = A(t)x$ where $A(t)$ is an $n \times n$ matrix whose elements a_{ij} are functions of time and $x(t)$ is a column vector of the n variables.

1.7.3 Gauss Jordan Elimination Method

Using Gauss Jordan elimination method, the coefficient matrix is reduced to a diagonal matrix. Hence, Gauss Jordan elimination method

$$[A \mid B] \xrightarrow{\text{Gauss Jordan Method}} [I \mid D]$$

If Gauss-Jordan elimination is applied on a square matrix, it can be used to obtain the inverse of the said matrix. This can be done by augmenting the square matrix with the identity matrix of the same dimensions and used to the following matrix operations:

$$[AI] \Rightarrow A^{-1}[AI] \Rightarrow [IA^{-1}]$$

Example 1.19 Find A^{-1} by GaussJordan elimination method: **MCA-06**

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Solution: By performing elementary row operations on the $[A|I]$ matrix until it reaches reduced row echelon form, the following is the final result:

$$[A \mid I] = \begin{bmatrix} 1 & 3 & 3 & | & 1 & 0 & 0 \\ 1 & 4 & 3 & | & 0 & 1 & 0 \\ 1 & 3 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 - R_1$ and $R_3 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 3 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$$

Applying $R_1 - 3R_2$, we get

$$\sim \begin{bmatrix} 1 & 0 & 3 & | & 4 & -3 & 0 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$$

Applying $R_1 - 3R_3$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 7 & -3 & -3 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & 0 & 1 \end{bmatrix}$$

Therefore the inverse of the given matrix is

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Example 1.20 Find all solutions of the system $\dot{x} = A(t)x$ with initial conditions $x(0) = [0, 1, -1]^T$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ te^{-t} & te^{-t} & 1 \end{bmatrix}$.

Solution: The solution is given in Theorem-1.13. Now find a fundamental solution matrix of the homogenous system $\dot{x}(t) = A(t)x(t)$. So the equations are $\dot{x}_1 = x_2, \dot{x}_2 = x_1$ and $\dot{x}_3 - x_3 = te^{-t}(x_1 + x_2)$.
From the first two equations, $x_1 = ae^t + be^{-t}$ and $x_2 = ae^t - be^{-t}$
The third equation now becomes $\dot{x}_3 - x_3 = 2at$ which has the general solution $x_3 = -2a(1+t) + ce^t$.
Hence a fundamental solution matrix is

$$X(t) = \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix},$$

By using Gauss-Jordan elimination Method to $X(t)$, we have,

$$X^{-1}(t) = \frac{1}{2}\begin{bmatrix} e^{-t} & e^{-t} & 0 \\ e^t & -e^t & 0 \\ 2(1+t)e^{-2t} & 2(1+t)e^{-2t} & 2e^{-t} \end{bmatrix}$$

and $X^{-1}(0) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$.

Thus the required solution is

$$\begin{aligned} x(t) &= \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \end{bmatrix} \end{aligned}$$

Hence the solution is

$$\begin{aligned} x_1(t) &= \frac{e^t}{2} - \frac{e^{-t}}{2} \\ x_2(t) &= \frac{e^t}{2} + \frac{e^{-t}}{2} \\ x_3(t) &= -(t+1). \end{aligned}$$

1.7.4 Structure of the solutions of $n-$ dimensional non-homogeneous linear systems

The general non-homogeneous, first order linear system of n dimensions is

$$\dot{x} = A(t)x + f(t), \tag{1.58}$$

where $A(t)$ is an $n \times n$ matrix whose elements a_{ij} are functions of time and $x(t)$ and $f(t)$ are the column vectors of the n variables.

The associate homogeneous system is

$$\dot{\Phi} = \mathbf{A}(t)\Phi. \tag{1.59}$$

The following properties are readily verified.

1. Let $x = x_p(t)$ be any solution of (1.58) (called a particular solution of the given system) and $\phi(t) = \phi_c(t)$ any solution of (1.59)(called the complementary function for the given system). Then $x_p(t) + \phi_c(t)$ is the general solution of (1.58).
2. Let $x_{p1}(t)$ and $x_{p2}(t)$ be any solutions of (1.58). Then $x_{p1}(t) - x_{p2}(t)$ is the solution of (1.59), i.e it is a complementary function.

Example 1.21 Find all solutions of the system

$$\begin{aligned} \frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= -y_1 + t \end{aligned}$$

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Solution: Here $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $f(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$ The corresponding homogenous system is

$\dot{\phi}_1(t) = \phi_2(t)$ and $\dot{\phi}_2(t) = -\phi_1(t)$ which is equivalent to $\ddot{\phi}_1(t)+\phi_1(t) = 0$. The linearly independent solutions $\phi_1 = \cos\, t$, $\sin\, t$ correspond respectively to $\phi_2 = -\sin\, t$, $\cos\, t$. Therefore, all solutions of the corresponding homogenous system are the linear combination of $\begin{pmatrix} \cos\, t \\ \sin\, t \end{pmatrix}, \begin{pmatrix} -\sin\, t \\ \cos\, t \end{pmatrix}$ which are given in matrix form by $\phi(t) = \begin{pmatrix} \cos\, t & -\sin\, t \\ \sin\, t & \cos\, t \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, where a_1, a_2 are arbitrary constants.

It is notice that $y_1 = t$ and $y_2 = 1$ is the particular solution of the given system. Therefore the general solution is

$$y(t) = \begin{pmatrix} \cos\, t & -\sin\, t \\ \sin\, t & \cos\, t \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix}$$

,

Theorem 1.17 The solution of the system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)ds,$$

where $\Phi(t)$ is any fundamental solution matrix of the corresponding homogeneous system $\dot{\Phi}(t) = \mathbf{A}(t)\Phi$

Proof. Let $\mathbf{x}(t)$ be the required solution, for which the following form is postulated

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\{\mathbf{x}_0 + \Phi(t)\} \tag{1.60}$$

The inverses of $\Phi(t)$ and $\Phi^{-1}(t_0)$ exist since, by Theorem 1.9, they are non-singular. Then by the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, or $\mathbf{x}_0 + \Phi(t_0)$ by (1.60) and so

$$\Phi(t_0) = 0.$$

To find the equation satisfied by $\Phi(t)$, substitute (1.60) into the equation, which becomes

$$\dot{\Phi}(t)\Phi^{-1}(t_0)\{\mathbf{x}_0 + \Phi(t)\} + \Phi(t)\Phi^{-1}(t_0)\dot{\Phi}(t) = \mathbf{A}(t)\Phi(t)\Phi^{-1}(t_0)\{\mathbf{x}_0 + \Phi(t)\} + \mathbf{f}(t).$$

Since $\Phi(t)$ is a solution matrix of the homogeneous equation, $\Phi(t) = \mathbf{A}(t)\Phi(t)$, and the previous equation then becomes

$$\Phi(t)\Phi^{-1}(t_0)\dot{\Phi}(t) = \mathbf{f}(t).$$

Therefore,

$$\dot{\Phi}(t) = \Phi(t_0)\Phi^{-1}(t)\mathbf{f}(t),$$

whose solution satisfying the initial condition is

$$\Phi(t) = \Phi(t_0)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)ds.$$

Therefore, by (1.60),

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)ds.$$

Example 1.22 Find all solutions of the system $\dot{\mathbf{x}} = \mathbf{A}(t)x + \mathbf{f}(t)$ with initial conditions $\mathbf{x}(0) = [0, 1, -1]^T$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ te^{-t} & te^{-t} & 1 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} e^t \\ 0 \\ 1 \end{bmatrix}$.

Solution: The solution is given in Theorem-1.17. Now find a fundamental solution matrix of the associated homogenous system $\dot{\Phi}(t) = A(t)\Phi(t)$. So the equations are $\dot{\Phi}_1 = \Phi_2$, $\dot{\Phi}_2 = \Phi_1$ and $\dot{\Phi}_3 - \Phi_3 = te^{-t}(\Phi_1 + \Phi_2)$. From the first two equations, $\Phi_1 = ae^t + be^{-t}$ and $\Phi_2 = ae^t - be^{-t}$
The third equation now becomes $\dot{\Phi}_3 - \Phi_3 = 2at$ which has the general solution $\Phi_3 = -2a(1+t)+ce^t$. Hence a fundamental solution matrix is

$$\Phi(t) = \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix},$$

$$\Phi^{-1}(t) = \frac{1}{2}\begin{bmatrix} e^{-t} & e^{-t} & 0 \\ e^t & -e^t & 0 \\ 2(1+t)e^{-2t} & 2(1+t)e^{-2t} & 2e^{-t} \end{bmatrix}$$

$$\text{and } \Phi^{-1}(0) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

Thus the required solution is

$$\begin{aligned}
x(t) &= \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&+ \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix} \frac{1}{2}\int_0^t \begin{bmatrix} e^{-s} & e^{-s} & 0 \\ e^s & -e^s & 0 \\ 2(1+s)e^{-2s} & 2(1+s)e^{-2s} & 2e^{-s} \end{bmatrix}\begin{bmatrix} e^s \\ 0 \\ 1 \end{bmatrix} ds \\
&= \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix} \frac{1}{2}\int_0^t \begin{bmatrix} 1 \\ e^{2s} \\ (4+2s)e^{-s} \end{bmatrix} ds \\
&= \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ -2(1+t) & 0 & e^t \end{bmatrix} \frac{1}{2}\begin{bmatrix} t \\ \frac{e^{2t}}{2} - \frac{1}{2} \\ (6 - 2(3+t)e^{-t} \end{bmatrix}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
x_1(t) &= (\frac{3}{4} + \frac{1}{2}t)e^t - \frac{3}{4}e^{-t} \\
x_2(t) &= (\frac{1}{4} + \frac{1}{2}t)e^t + \frac{3}{4}e^{-t} \\
x_3(t) &= 3e^t - t^2 - 3t - 4.
\end{aligned}$$

Lemma 1.1 Let $\Phi(t)$ be any fundamental matrix of the system $\dot{\Phi} = \mathbf{A}\Phi$, $\mathbf{A}$ is constant matrix. Then for any two parameters s , t_0,

$$\Phi(t)\Phi^{-1}(s) = \Phi(t - s + t_0)\Phi^{-1}(t_0).$$

In particular,

$$\Phi(t)\Phi^{-1}(s) = \Phi(t - s)\Phi^{-1}(0).$$

Proof. Since $\dot{\Phi} = \mathbf{A}\Phi$, if we define $\mathbf{U}(t) = \Phi(t)\Phi^{-1}(s)$, then $\dot{\mathbf{U}(t)} = \mathbf{AU}(t)$, and $\mathbf{U}(\mathbf{s}) = \mathbf{I}$.
Now consider $\mathbf{V}(t) = \Phi(t - s + t_0)\Phi^{-1}(t_0)$. Then $\dot{\mathbf{V}(t)} = \mathbf{AV}(t)$. (for since $\mathbf{A}$ is constant, $\Phi(t)$ and $\Phi(t - s + t_0)$ satisfy the same equation), and $\mathbf{V}(s) = \mathbf{I}$.
Therefore, the corresponding columns of $\mathbf{U}$ and $\mathbf{V}$ satisfy the same equation with the same initial conditions, and are therefore identical by the Uniqueness Theorem.

Theorem 1.18 Let $\mathbf{A}$ be a constant matrix. The solution of the system $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{f}(t)$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t - s + t_0)\Phi^{-1}(t_0)\mathbf{f}(s)ds,$$

where $\Psi(t)$ is any fundamental matrix satisfying $\Psi(t_0) = \mathbf{I}$, then

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \int_{t_0}^{t} \Psi(t - s)\mathbf{f}(s)ds.$$

Proof. The Theorem 1.18 is obtained by applying the Lemma 1.1 to Theorem 1.17.

Example 1.23 express the solution of the second order equation $\ddot{x} - x = f(t)$ with $x(0) = 0$, $\dot{x}(0) = 1$ as an integral.

Solution: An equivalent first order differential equation is

$$\dot{x} = y, \quad \dot{y} = x + f(t),$$

and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{f}(t).$$

Since the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$, $\lambda_2 = -1$ and the corresponding eigenvectors are $\mathbf{r}_1 = [1, 1]^T$ and $\mathbf{r}_2 = [1, -1]^T$, a fundamental matrix for the homogeneous system is

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}.$$

Then, following Theorem 1.18

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \Phi^{-1}(0) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}, \end{pmatrix}$$

$$\text{and}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}, \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int\limits_0^t \begin{pmatrix} e^{t-s} & e^{-t+s} \\ e^{t-s} & -e^{-t+s} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}, \end{pmatrix}\begin{pmatrix} 0 \\ f(t) \end{pmatrix} ds$$

$$= \begin{pmatrix} \sinh t \\ \cosh \end{pmatrix} + \int\limits_0^t \begin{pmatrix} f(t)\sinh(t-s) \\ f(t)\cosh(t-s) \end{pmatrix} ds.$$

1.7.5 Structure of the solutions of $n-$ dimensional non-homogeneous linear systems with constant coefficients

The general non-homogeneous, first order linear system of n dimensions is

$$\dot{x}(t) = Ax(t) + f(t). \tag{1.61}$$

where A is an $n \times n$ matrix whose elements a_{ij} are real constants, $x(t)$ and $f(t)$ are the column vectors of the n variables.
The complementary solution of the homogeneous system $\dot{x}(t) = Ax(t)$ has been obtained as $x(t) = X(t)C$, where $X(t)$ is a fundamental matrix, whose columns are linearly independent and each is a solution of the homogeneous system, i.e., $\dot{X}(t) = AX(t)$, and C is an n-dimensional constant vector.
Applying the method of variation of parameters, vary the constant vector C in the complementary solution $x(t) = X(t)C$ to make it a vector of functions of t, i.e., $C = c(t)$. Thus a particular solution is assumed to be of the form

$$x(t) = X(t)c(t).$$

Differential with respect to t yields

$$\begin{aligned}
&\dot{x}(t) = \dot{X}(t)c(t) + X(t)\dot{c}(t) = Ax(t) + f(t) \\
&\text{Substituting } X'(t) = AX(t) \text{ and } x(t) = X(t)c(t) \text{ yields} \\
&AX(t)c(t) + X(t)c'(t) = AX(t)c(t) + f(t), \\
&X(t)c'(t) = f(t) \Rightarrow c'(t) = X^{-1}(t)f(t)
\end{aligned}$$

Integrating with respect to t gives

$$c(t) = C + \int X^{-1}(t)f(t)dt$$

Hence, the general solution is given by

$$x(t) = X(t)c(t) = X(t)\Big\{C + \int X^{-1}(t)f(t)dt\Big\}$$

For the nonhomogeneous system $x'(t) = Ax(t) + f(t)$ with the initial condition $x(t_0) = x(_0)$,the general solution can be written as

$$x(t) = X(t)\Big\{C + \int_{t_0}^{t} X^{-1}(t)f(t)dt\Big\}$$

with $x(t_0) = X(t_0)C \Rightarrow C = X^{-1}(t_0)x(t_0)$, which yields. So, to find a particular solution using the method of variation of parameters, one must evaluate the inverse $X^{-1}(t)$ of a fundamental matrix $X(t)$. Finally the general solution is

$$x(t) = X(t)\Big\{X^{-1}(t_0)x(t_0) + \int_{t_0}^{t} X^{-1}(t)f(t)dt\Big\}.$$

Example 1.24 Find the general solution for the system

$$\begin{aligned}\frac{dy_1}{dt} &= -3y_1 - 4y_2 + 2e^{-t}\\ \frac{dy_2}{dt} &= y_1 + y_2\end{aligned}$$

Solution: In the matrix form, the system of differential equations can be written as $\dot{\mathbf{y}}(t) = A\mathbf{y}(\mathbf{t}) + f(t)$ with

$$y(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ A = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix},\ f(t) = \begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1,\ -1.$$

Hence, $\lambda = -1$ is an eigenvalue of multiplicity 2. The eigenvector equation for $\lambda = -1$ is

$$(A - \lambda I)v_1 = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_{11} + 2v_{21} = 0.$$

Taking $v_{21} = -1$, then $v_{11} = -2v_{21} = 2$,

$$\therefore\ v_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The second linearly independent eigenvector does not exist. Hence, matrix A is imperfective and a complete basis of eigenvectors is obtained by including a generalized eigenvector:

$$(A - \lambda I)v_2 = v_1 \Rightarrow \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow v_{21} + 2v_{22} = -1.$$

Taking $v_{22} = -1$, then $v_{11} = -1 - 2v_{21} = 1$,

$$\therefore\ v_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Two linearly independent solutions are

$$\mathbf{y_1} = e^{\lambda t}v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}e^{-t},\ \mathbf{y_2} = e^{\lambda t}(v_1 t + v_2) = \Big\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}t + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\Big\}e^{-t}$$

The fundamental matrix is

$$\mathbf{Y(t)} = [y_1,\ y_2] = \begin{pmatrix} 2e^{-t} & (2t+1)e^{-t} \\ -e^{-t} & -(t+1)e^{-t} \end{pmatrix},\ \det(y) = -2e^{-2t}$$

and its inverse is obtained as

$$\mathbf{Y^{-1}(t)} = [y_1,\ y_2] = \begin{pmatrix} (t+1)e^t & (2t+1)e^t \\ -e^t & -2e^t \end{pmatrix}.$$

It is easy to evaluate

$$\begin{aligned}\int Y^{-1}(t)f(t)dt &= \int \begin{pmatrix} (t+1)e^t & (2t+1)e^t \\ -e^t & -2e^t \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix} dt \\ &= \int \begin{pmatrix} 2(t+1) \\ -2 \end{pmatrix} dt = \begin{pmatrix} t^2+2t \\ -2t \end{pmatrix}\end{aligned}$$

The general solution is

$$\begin{aligned} y(t) &= y(t)\Big\{C + \int Y^{-1}(t)f(t)dt\Big\} = \begin{pmatrix} 2e^{-t} & (2t+1)e^{-t} \\ -e^{-t} & -(t+1)e^{-t} \end{pmatrix}\begin{pmatrix} c_1+t^2+2t \\ c_2-2t \end{pmatrix} \\ y_1(t) &= e^{-t}[-2t^2+2(c_2+1)t+(2c_1+c_2)],\ y_2(t) = e^{-t}[t^2-c_2t-(c_1+c_2)]. \end{aligned}$$

1.8 Solution of Simultaneous Equation of Type-II

Simultaneous Linear Differential Equations of Type-II is of the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{1.62}$$

where P, Q and R are functions of x, y, z.
By the solution of the equation (1.62), we mean to find a solution of the form $\phi(x,y,z) = 0$ and $\psi(x,y,z) = 0$ where ϕ and ψ are two independent integrals of the given equations (1.62) viz. $c_1\phi + c_2\psi = 0$ is possible only when two arbitrary constants c_1 and c_2 are zero individually.

1.8.1 Methods of Solution of Equations of Type-II

The equation of the type

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

can be solved by using different techniques depending on the nature of P, Q and R.

1.8.2 Method-I

When P, Q and R such function of x, y, z that any two parts of the equations (1.62), when considered separately, can be solved by the method of separation of variables. Then considering two parts separately we get a relation between two variables, say $u = 0$ and then considering another two parts we get another relation among the variables like $v = 0$ and finally $u = 0$, $v = 0$ will give the general solution of the system.

Example 1.25 Solve: $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$

Solution : We have

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy} \tag{1.63}$$

From the first two ratio of (1.63), we get $xdx - ydy = 0$. Then integrating we get

$$\text{i.e, } x^2 - y^2 = c_1,\ c_1 \text{being an integrating constant} \tag{1.64}$$

Again from last two parts we get, $zdz - ydy = 0$ and integrating, we have $z^2 - y^2 = c_2$, c_2 being an integrating constants. Eliminating c_1, c_2, we get the required solution(surface) as $\phi(x^2 - y^2, z^2 - y^2) = 0$.

1.8.3 Method-II

If P, Q and R are such, that considering two parts of the equation (1.62), like method-I, a relation between two variables can be found and using this relation another two parts can be integrated to get other relation connecting the variable and ultimately we get two independent relation to represent the general solution of (1.62).

Example 1.26 Solve: $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$.

Solution : We have

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2} \tag{1.65}$$

From the first two ratio of (1.65), we have $\frac{dx}{x} = \frac{dy}{y}$. Then integrating, we get

$$x = c_1 y, \tag{1.66}$$

c_1 being an integrating constant. Again, from last two parts we get
$\frac{dy}{y^2} = \frac{dz}{xyz - 2x^2} \Rightarrow \frac{dy}{y^2} = \frac{dz}{c_1 y^2 z - 2c_1^2 y^2}$ (using (1.66)) and then integrating, we get

$$\begin{aligned} & c_1 y = \log(z - 2c_1) + c_2,\ c_2 \text{being an integrating constants} \\ \Rightarrow \quad & x = \log(z - \frac{2x}{y}) + c_2 \ [\text{since}\ x = c_1 y] \end{aligned}$$

Hence the required solution is $\log(z - \frac{2x}{y}) - x = \phi(\frac{x}{y})$.

1.8.4 Method-III

If P_1, Q_1 and R_1 be such function of (x, y, z) that when we write the given equations as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1}$$
$$\text{then}\ PP_1 + QQ_1 + RR_1 = 0$$

So we may write $P_1dx + Q_1dy + R_1dz = 0$. Now if these P_1, Q_1 and R_1 also be such that $P_1dx + Q_1dy + R_1dz = 0$ can be integrated to get a function $\phi(x, y, z) = 0$. If another set of such (P_1, Q_1, R_1) can be found, we get another integral $\psi(x, y, z) = 0$ and these two integrals together will give the solution of (1.62).

Example 1.27 Solve: $\frac{dx}{3y-2z} = \frac{dy}{z-3x} = \frac{dz}{2x-y}$

Solution: The given equation

$$\frac{dx}{3y-2z} = \frac{dy}{z-3x} = \frac{dz}{2x-y} \tag{1.67}$$

can be written by Method-III as

$$\begin{aligned} dx + 2dy + 3dz &= 0 \\ xdx + ydy + zdz &= 0 \end{aligned}$$

Which integrate to two families of surfaces

$$x + 2y + 3z = c_1 \tag{1.68}$$
$$x^2 + y^2 + z^2 = c_2 \tag{1.69}$$

Where c_1, c_2 are two arbitrary constants. Then the required general solution is given by (1.68) and (1.69) or $\phi(x^2 + y^2 + z^2, x + 2y + 3z) = 0$

1.8.5 Method-IV

If P_1, Q_1 and R_1 be such function of (x, y, z) that

$$\frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1} = \frac{d(PP_1 + QQ_1 + RR_1)}{PP_1 + QQ_1 + RR_1}$$

Then combing this with one of the ratio $\frac{dx}{P}$ or $\frac{dy}{Q}$ or $\frac{dz}{R}$, we may get one function $\phi(x, y, z) = 0$ and if otherwise we can get another functional relation $\psi(x, y, z) = 0$, then combining these two, we get the solution of the equation (1.62).

Example 1.28 Solve

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)}$$

Solution : We have

$$\begin{aligned} & \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \\ \Rightarrow \quad & \frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)} \end{aligned} \tag{1.70}$$

Taking first two parts of (1.70), we get

$$\begin{aligned} & \frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} \\ \Rightarrow \quad & x^2dx + y^2dy = 0 \end{aligned}$$

Integrating, we get

$$x^3 + y^3 = c_1, \tag{1.71}$$

where c_1 being an arbitrary constants. Again, each part of (1.70) is equal to

$$\frac{dx - dy}{y^2(x-y) - x^2(y-x)} = \frac{d(x-y)}{(x-y)(x^2+y^2)}$$

Now combining this with the third part of (1.70), we get

$$\begin{aligned} \frac{d(x-y)}{(x-y)(x^2+y^2)} &= \frac{dz}{z(x^2+y^2)} \\ \Rightarrow \qquad \frac{d(x-y)}{x-y} &= \frac{dz}{z} \end{aligned}$$

Integrating, we get, $\log(x-y) - \log z = \log c_2$, where c_2 being an arbitrary constants.

$$\text{i.e}, \frac{x-y}{z} = c_2 \tag{1.72}$$

So the general integral is given, using (1.71) and (1.72) as $\phi(x^3 + y^3, \frac{x-y}{z}) = 0$.

1.8.6 Geometrical Interpretation of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

It is known from geometry that the direction cosines of the tangent to a curve are given by $(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds})$. Thus the direction cosines of this tangent are proportional to dx, dy and dz. Again from the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, we see that dx, dy and dz are proportional to P, Q and R. Thus (P, Q, R) are the corresponding direction rations of the tangents to the curves at (x, y, z). Thus geometrically the above differential equations represents a system of curves in space such that the direction cosines of the tangent to these curves at any point (x, y, z) are proportional to (P, Q, R).

1.9 Worked Out Examples

Example 1.29 Find the first order simultaneous differential equations of the system

$$\frac{d^3x}{dt^3} - 2\frac{d^2x}{dt^2} - \frac{dx}{dt} + 2x = e^t$$

Solution: Write as the equivalent first order system

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = z, \ \frac{dz}{dt} = -2x + y + 2z + e^t$$

Example 1.30 Let $P_1, P_2, \cdots P_n$ be n constants, then show that the differential equation $\frac{d^n x(t)}{dt^n} + P_1 \frac{d^{n-1}x(t)}{dt^{n-1}} + P_2 \frac{d^{n-2}x(t)}{dt^{n-2}} + \cdots + P_n x(t) = 0$ is equivalent to the system

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= x_2(t), \text{ where } x_1(t) = x(t) \\
\frac{dx_2(t)}{dt} &= x_3(t) \\
\vdots \quad \vdots & \quad \vdots \\
\frac{dx_{n-1}(t)}{dt} &= x_n(t) \\
\frac{dx_n(t)}{dt} &= -P_n x_1(t) - P_{n-1} x_2(t) - \cdots - P_1 x_n(t).
\end{aligned}$$

Show that the equation for the eigenvalues is

$$\lambda^n + P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \cdots + P_n = 0.$$

Proof. The given differential equation $\frac{d^n x(t)}{dt^n} + P_1 \frac{d^{n-1}x(t)}{dt^{n-1}} + P_2 \frac{d^{n-2}x(t)}{dt^{n-2}} + \cdots + P_n x(t) = 0$ can be written as

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= x_2(t), \text{ where } x_1(t) = x(t) \\
\frac{dx_2(t)}{dt} &= x_3(t) \\
\vdots \quad \vdots & \quad \vdots \\
\frac{dx_{n-1}(t)}{dt} &= x_n(t) \\
\frac{dx_n(t)}{dt} &= -P_n x_1(t) - P_{n-1} x_2(t) - \cdots - P_1 x_n(t).
\end{aligned}$$

The matrix of coefficients for the above system is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -P_n & -P_{n-1} & -P_{n-2} & -P_{n-3} & \cdots & -P_1 \end{bmatrix}.$$

The eigenvalues are given by

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -P_n & -P_{n-1} & -P_{n-2} & -P_{n-3} & \cdots & -P_1 - \lambda \end{vmatrix} = 0.$$

Let $D_n(\lambda)$ denoted the determinant in the previous equation. Then expansion by row 1 leads to

$$D_n(\lambda) = -\lambda D_{n-1}(\lambda) + (-1)^n P_n. \tag{1.73}$$

For decreasing n, we have

$$D_{n-1}(\lambda) = -\lambda D_{n-2}(\lambda) + (-1)^{n-1}P_{n-1}, \tag{1.74}$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$D_2(\lambda) = -\lambda D_1(\lambda) + P_2. \tag{1.75}$$

where $D_1(\lambda) = -P_1 - \lambda$. Now eliminating $D_{n-1}(\lambda)$, $D_{n-2}(\lambda), \cdots$ from equations (1.73) through (1.75) by multiplying successive equations by $-\lambda$, $+\lambda$ and so on, and adding them. The result is

$$D_n(t) = (-1)^n(P_n + P_{n-1}\lambda + \cdots + P_1\lambda^{n-1} + \lambda^n).$$

The required result follows by equating $D_n(\lambda) = 0$. Therefore, the equation for the eigenvalues is $\lambda^n + P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots + P_n = 0$.

Example 1.31 Solve $\frac{dx}{dt} + 3x + y = e^t$, $\frac{dy}{dt} - x + y = e^{2t}$

Solution: Writing D for $\frac{d}{dt}$, the equations are

$$(D+3)x + y = e^t \tag{1.76}$$
$$\text{and} \qquad (D+1)y - x = e^{2t} \tag{1.77}$$

Putting the value of $y = e^t - (D+3)x$ in (1.77), we get

$$\begin{aligned} & (D+1)\{e^t - (D+3)x\} - x = e^{2t} \\ \Rightarrow\ & (D+1)e^t - (D+1)(D+3)x - x = e^{2t} \\ \Rightarrow\ & e^t + e^t - (D^2 + 4D + 3 + 1)x = e^{2t} \\ \Rightarrow\ & (D^2 + 4D + 4)x = 2e^t - e^{2t} \end{aligned} \tag{1.78}$$

Let $y(x) = e^{mt}$ (m being a constant) be a trial solution of the corresponding homogenous differential equation of (1.78). Then its auxiliary equation is

$$\begin{aligned} & m^2 + 4m + 4 = 0 \\ \Rightarrow\ & m = -2, -2 \end{aligned}$$

The complementary function of the equation (1.78) is

$$C.F = (A + Bt)e^{-2t}, \text{ where } A \text{ and } B \text{ are arbitrary constants.}$$

The particular integral of (1.78) is

$$\begin{aligned} P.I &= \frac{1}{D^2 + 4D + 4}(2e^t - e^{2t}) \\ &= \frac{2}{D^2 + 4D + 4}e^t - \frac{1}{D^2 + 4D + 4}e^{2t} \\ &= \frac{2e^t}{1^2 + 4.1 + 4} - \frac{e^{2t}}{2^2 + 4.2 + 4} \\ &= \frac{2e^t}{9} - \frac{e^{2t}}{16} \end{aligned}$$

Therefore the general solution of the equation (1.78) is

$$x = (A + Bt)e^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{16}$$

Putting these value of x in (1.76), we get

$$\begin{aligned}
y &= e^t - (D+3)x \\
&= e^t - (D+3)\Big((A+Bt)e^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{16}\Big) \\
&= e^t - \frac{d}{dt}\Big((A+Bt)e^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{16}\Big) - 3\Big((A+Bt)e^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{16}\Big) \\
&= e^t - \Big(-2(A+Bt)e^{-2t} + Be^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{8}\Big) - 3\Big((A+Bt)e^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{16}\Big) \\
&= -(A+B+Bt)e^{-2t} + \frac{e^t}{9} + \frac{5}{16}e^{2t}.
\end{aligned}$$

Therefore the solution of the given simultaneous linear equation is given by

$$y = -(A+B+Bt)e^{-2t} + \frac{e^t}{9} + \frac{5}{16}e^{2t}.$$

$$\text{and} \qquad x = (A+Bt)e^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{16}.$$

Example 1.32 Solve $\frac{dx}{dt} + 3x + y = e^t$, $\frac{dy}{dt} - x + y = e^{2t}$

Solution: Writing D for $\frac{d}{dt}$, the equations are

$$(D+3)x + y = e^t \tag{1.79}$$

$$\text{and} \qquad -x + (D+1)y = e^{2t} \tag{1.80}$$

Let the complementary solutions of the given system of linear differential equation (1.80) be $x_c(t), y_c(t)$. Then x_c, y_c are satisfied the homogenous differential equations of (1.79) and (1.80) *i.e.*

$$\phi(D)x_c(t) = 0,\ \phi(D)y_c(t) = 0$$

where $\phi(D)$ is determinant of the coefficient matrix of the given system of linear differential equations (1.79)-(1.80) i.e,

$$\phi(D) = \begin{vmatrix} D+3 & 1 \\ -1 & D+1 \end{vmatrix}$$

Hence the unknowns $x_c(t), y_c(t)$, all have the same characteristic equation $\phi(\lambda) = 0$ and, as a result, the same form of complementary solutions. Now,

$$\phi(\lambda) = \begin{vmatrix} \lambda+3 & 1 \\ -1 & \lambda+1 \end{vmatrix} = (\lambda+2)^2$$

So, $\phi(\lambda) = 0, \Rightarrow \lambda = -2, -2$. Hence, they have the same complementary solutions given by $x_c = (A+Bt)e^{-2t}$ and $y_c = (C+Dt)e^{-2t}$.

To find the particular solution of the given system of linear differential equation (1.80), we have,

$$\Delta_x(t) = \begin{vmatrix} e^t & 1 \\ e^{2t} & D+1 \end{vmatrix} = 2e^t - e^{2t},$$

$$\Delta_y(t) = \begin{vmatrix} D+3 & e^t \\ -1 & e^{2t} \end{vmatrix} = 5e^{2t} + e^t,$$

$$x_p(t) = \frac{\Delta_x(t)}{\phi(D)} = \frac{2e^t - e^{2t}}{(D+2)^2} = \frac{2e^t}{9} - \frac{e^{2t}}{16},$$
$$y_p(t) = \frac{\Delta_y(t)}{\phi(D)} = \frac{5e^{2t} + e^t}{(D+2)^2} = \frac{e^t}{9} + \frac{5e^{2t}}{16}.$$

The general solutions are

$$x(t) = x_c(t) + x_p(t) = (A + Bt)e^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{16} \quad (1.81)$$
$$y(t) = y_c(t) + y_p(t) = (C + Dt)e^{-2t} + \frac{e^t}{9} + \frac{5e^{2t}}{16} \quad (1.82)$$

Since $\phi(D) = 0$ is a polynomial of degree 2 in D, the general solutions should contain only two arbitrary constants. The two extra constants can be eliminated by substituting equations into either (1.81) or (1.82). Substitute the equations (1.81) and (1.82) in (1.80) to eliminate the two extra constants, we get, $C = -A - B$ and $D = -B$. Then the general solutions become

$$x(t) = (A + Bt)e^{-2t} + \frac{2e^t}{9} - \frac{e^{2t}}{16},$$
$$y(t) = -(A + B + Bt)e^{-2t} + \frac{e^t}{9} + \frac{5}{16}e^{2t}$$

Note:(Alternative Method) The two extra constants C, D can be eliminated by substituting the complementary solutions $x_c(t)$, $y_c(t)$ into either the homogeneous equation of (1.79) or (1.80). So, substitute the complementary solutions $x_c(t)$, $y_c(t)$ in $(D+3)x + y = 0$ i.e., $(D+3)x_c(t) + y_c(t) = 0$ to eliminate the two extra constants, we get, $C = -A - B$ and $D = -B$.

Example 1.33 Find the solution for the system

$$y_1' = 4y_1 + y_2$$
$$y_2' = -y_1 + 2y_2.$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 2$.

Solution:

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$$

and there is only one eigenvector,

$$\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

with eigenvalue $\lambda = 3$. The general solution is

$$\mathbf{y} = c_1 \mathbf{v} e^{\lambda t} + c_2(t\mathbf{v} + \mathbf{u})e^{\lambda t}$$

where $\mathbf{u}$ satisfies

$$(A - \lambda \mathbf{I})\,\mathbf{u} = \mathbf{v} \text{ (See example 1.16)}$$

and so, in this case,

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and a solution to this is

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

and so the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \left[t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{3t}$$

Now, putting $t = 0$ we get

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

and, hence,

$$\begin{aligned} 3 &= -c_1 - c_2 \\ 2 &= c_1 \end{aligned}$$

also $c_2 = 1$ and $c_2 = -5$ giving

$$\mathbf{y} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} - 5 \left[t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{3t}$$

or

$$\begin{aligned} y_1 &= (3 + 5t)e^{3t} \\ y_2 &= (2 - 5t)e^{3t} \end{aligned}$$

Example 1.34 Find the solution for the system

$$\begin{aligned} \frac{dy_1}{dt} &= -3y_1 + 2y_2 \\ \frac{dy_2}{dt} &= -2y_1 + 2y_2 \end{aligned}$$

Solution: This equation is $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}$$

We can find the eigenvalues, the characteristic equation is

$$\begin{vmatrix} -3 - \lambda & 2 \\ -2 & 2 - \lambda \end{vmatrix} = (\lambda + 3)(\lambda - 2) + 4 = \lambda^2 + \lambda - 2 = 0$$

so that $\lambda_1 = 1$ and $\lambda_2 = -2$.

Next, we need the eigenvectors. First, λ_1:

$$\begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

so $-3a + 2b = a$ or $b = 2a$, hence, choosing $a = 1$ we get

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For λ_2:

$$\begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -2\begin{pmatrix} a \\ b \end{pmatrix}$$

so $-3a + 2b = -2a$ giving $a = 2b$, choosing $b = 1$ gives

$$\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now, in general the solution is

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}$$

so, here,

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t}$$

Example 1.35 Find the general solution to

$$y' - 2y = -t$$

Solution: This follows from the general solution to

$$y' + ry = f(t) \tag{1.83}$$

which is

$$y = Ce^{-rt} + e^{-rt} \int e^{rt} f dt \tag{1.84}$$

so here $r = -2$ and $f(t) = -t$ so, using integration by parts

$$\begin{aligned} y &= Ce^{2t} - e^{2t} \int te^{-2t} dt \\ &= Ce^{2t} - e^{2t} \left\{ -\frac{1}{2} te^{-2t} + \frac{1}{2} \int e^{-2t} dt \right\} \\ &= Ce^{2t} - e^{2t} \left\{ -\frac{1}{2} te^{-2t} - \frac{1}{4}(e^{-2t}) \right\} \\ &= Ce^{2t} + \frac{t}{2} + \frac{1}{4} \end{aligned} \tag{1.85}$$

Example 1.36 Solve: $\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$

Solution: We have

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}.$$

Gives rise

$$\begin{aligned} \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} &= 0 \\ \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} &= 0 \end{aligned}$$

Which integrate to two families of surfaces

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_1$$
$$\Rightarrow \quad \log x+\log y+\log z=\log c_2 \Rightarrow xyz=c_2$$

Where c_1, c_2 are two arbitrary constants. Therefore the general integral is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\phi(xyz)$.

Example 1.37 Solve: $\frac{dx}{x(y^2+z^2)}=\frac{dy}{y(x^2+z^2)}=\frac{dz}{z(x^2-y^2)}$.

Solution : We have

$$\frac{dx}{x(y^2+z^2)}=\frac{dy}{y(x^2+z^2)}=\frac{dz}{z(x^2-y^2)} \tag{1.86}$$

Each ratio of(1.86), $\frac{\frac{dx}{x}-\frac{dy}{y}+\frac{dz}{z}}{y^2+z^2-x^2-z^2+x^2-y^2}=\frac{\frac{dx}{x}-\frac{dy}{y}+\frac{dz}{z}}{0} \Rightarrow \frac{dx}{x}-\frac{dy}{y}+\frac{dz}{z}=0$ and integrating we get, $\log x-\log y+\log z=\log c_1$.

$$\text{i.e., } \frac{xz}{y}=c_1, \tag{1.87}$$

where c_1 being integrating constant. Again, each ratio of (1.86) is given by

$$\frac{xdx-ydy-zdz}{x^2(y^2+z^2)-y^2(x^2+z^2)-z^2(x^2-y^2)}=\frac{xdx-ydy-zdz}{0} \Rightarrow xdx-ydy-zdz=0$$

Integrating, we get,

$$x^2-y^2-z^2=c_2 \tag{1.88}$$

c_2 being an integrating constant. Then the equations (1.87) and (1.88) constitute the general solution of the equation as $\phi(x^2-y^2-z^2, \frac{xz}{y})=0$.

Example 1.38 Solve: $\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)}$ which contains the straight line $x+y=0,\ z=1$.

Solution : We have

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)} \tag{1.89}$$

Each ratio of(1.89) is $\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2+z-x^2-z+x^2-y^2}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$ i.e $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$. Then integrating, we get $\log(xyz)=\log c_1$.

$$\text{i.e., } xyz=c_1, \tag{1.90}$$

where c_1 being integrating constant. Again, each ratio of (1.89) is given by

$$\frac{xdx+ydy-dz}{x^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)}=\frac{xdx+ydy-dz}{0} \Rightarrow xdx+ydy-dz=0$$

Integrating, we get,

$$x^2+y^2-2z=c_2 \tag{1.91}$$

c_2 being an integrating constant. Here the integral surface contains the straight line $x+y=0,\ z=1$. Now putting $x=t,\ y=-t$ and $z=1$ in (1.90) and (1.91), we get, $c_1=-t^2$ and $c_2=2t^2-2$. Eliminating t between the two , we get, $2c_1+c_2+2=0$. Putting the value of c_1 and c_2 from (1.90) and (1.91), we get the required integral surface as $x^2+y^2+2xyz-2z+2=0$.

Example 1.39 Solve: $\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}$ which passes through the circle $x^2+y^2 = 2x,\ z = 0$.

Solution : We have

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \tag{1.92}$$

From the first and last part of (1.92), we get, $\frac{dx}{2(z-3)} = \frac{dz}{2x-3} \Rightarrow (2x-3)dx - 2(z-3)dz = 0$. Then integrating, we get

$$x^2 - 3x - z^2 + 6z = c_1, \tag{1.93}$$

c_1 being an integrating constant. Again each ration of (1.92) is equal to $\frac{\frac{1}{2}dx+ydy-dz}{y(z-3)+y(2x-z)-y(2x-3)} = \frac{\frac{1}{2}dx+ydy-dz}{0} \Rightarrow \frac{1}{2}dx + ydy - dz = 0$ and again integrating we get,

$$x + y^2 - 2z = c_2, \tag{1.94}$$

c_2 being an integrating constant. Adding (1.93) and (1.94), we get, $x^2 + y^2 - 2x - z^2 + 4z = c_1 + c_2 \Rightarrow c_1 + c_2 = 0$, (using $z = 0$, $x^2 + y^2 = 2x$). After that putting the value of c_1 , c_2 from (1.93) and (1.94), we get the required integral surface as $x^2 + y^2 - z^2 - 2x + 4z = 0$.

1.10 Multiple Choice Questions

1. Consider a system of first order differential equations $\dot{x}(t) = x(t) + y(t),\ \dot{y}(t) = -y(t)$. The solution space is spanned by
   (a) $[0, e^{-t}]^T$ and $[e^t, 0]^T$ (b) $[e^t, 0]^T$ and $[\cosh t, e^{-t}]^T$ [NET(Dec.)MA-2017]
   (c) $[e^{-t}, -2e^{-t}]^T$ and $[\sinh t, e^{-t}]^T$ (d) $[e^t, 0]^T$ and $[e^t - \frac{e^{-t}}{2}, e^{-t}]^T$
   Ans. (c) and (d).
   Hint. Eigenvalues $\lambda = -1, 1$, and eigenvectors $V_1 = (1, -2)^T$, $V_2 = (1, 0)^T$ so general solution is $[x, y]^T = k_1[e^t, 0]^T + k_2[e^{-t}, -2e^{-t}]^T$. Taking $k_2 = 1, k_1 = 0$; $k_1 = \frac{1}{2}, k_2 = -\frac{1}{2}$ and $k_1 = 1, k_2 = 0; k_1 = 1, k_2 = \frac{1}{2}$.
2. Let y_1 and y_2 be twice differentiable functions on a interval I satisfying the differential equations $y_1' - y_1 - y_2 = e^x$ and $2y_1' + y_2' - 6y_1 = 0$. Then $y_1(x)$ is
   (a) $c_1e^{-2x} + c_2e^{3x} - \frac{1}{4}e^x$ (b) $c_1e^{2x} + c_2e^{-3x} - \frac{1}{4}e^x$ (c) $c_1e^{-2x} + c_2e^{-3x} - \frac{1}{8}e^x$
   (d) $c_1e^{3x} + c_2e^{-2x} - \frac{1}{4}e^x$ [JAM MA-2008]
   Ans. (b)
3. Consider the system of ODE $\frac{dY}{dx} = AY,\ Y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ where $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$
   NET(MS): (June)2012
   (a) $y_1(x) \to \infty$ and $y_2(x) \to 0$ as $x \to \infty$
   (b) $y_1(x) \to 0$ and $y_2(x) \to 0$ as $x \to \infty$
   (c) $y_1(x) \to \infty$ and $y_2(x) \to -\infty$ as $x \to -\infty$
   (d) $y_1(x) \to -\infty$ and $y_2(x) \to \infty$ as $x \to -\infty$
   Ans. (a) and (c).
   Hint. Here the eigenvectors are $[1, 0]^T$ and $[1, -1]^T$ corresponding to the eigenvalues -1, 1. So the general solutions are $y_1(x) = Ae^x + Be^{-x}$ and $y_2(x) = -Be^{-x}$. Using the given initial condition $Y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, we have $y_1(x) = e^x + e^{-x}$ and $y_2(x) = -e^{-x}$. Hence the result.

4. Let $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ satisfy $\frac{dY}{dt} = AY$, $t > 0$, $Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where A is a 2×2 constant matrix with real entries satisfying $trace\, A = 0$ and $det\, A > 0$. Then $y_1(t)$ and $y_2(t)$ both are **NET(MS): (Dec.)2012**
(a) monotonically decreasing functions of t.
(b) monotonically increasing functions of t.
(c) oscillating functions of t.
(d) constant functions of t.
Ans. (c).

5. Consider the first order system of linear equations $\frac{dX}{dt} = AX$ where $A = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$ and $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Then **NET(MS): (Dec.)2011**
(a) the coefficient matrix A has a repeated eigenvalue $\lambda = 1$.

(b) there is only one linearly independent eigenvector $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(c) the general solution of the ODE is $(aX_1 - bX_2)e^t$, where a and b are arbitrary constants and $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $X_2 = \begin{pmatrix} t \\ \frac{1}{2} - t \end{pmatrix}$.
(d) the vectors X_1 and X_2 in the option (c) given above are linearly independent
Ans. (a), (b), (c) and (d).

6. The general solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of the system

$$\dot{x} = -x + 2y$$
$$\dot{y} = 4x + y$$

is given by **GATE(MA)-04**
A) $\begin{cases} C_1e^{3t} - C_2e^{-3t} \\ 2C_1e^{3t} + C_2e^{-3t} \end{cases}$ B) $\begin{cases} C_1e^{3t} \\ C_2e^{-3t} \end{cases}$ C) $\begin{cases} C_1e^{3t} + C_2e^{-3t} \\ 2C_1e^{3t} + C_2e^{-3t} \end{cases}$ D) $\begin{cases} C_1e^{3t} - C_2e^{-3t} \\ -2C_1e^{3t} + C_2e^{-3t} \end{cases}$
Ans. A)

7. Let $A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$, $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ and $|x(t)| = \sqrt{(x_1^2(t) + x_2^2(t) + x_3^2(t))}$. Then any solution of the first order system of the ordinary differential equation **NET(JUNE)-16**

$$\dot{x}(t) = Ax(t), \ \ \mathbf{x(0) = 0}$$

satisfies
(a) $\lim_{t\to\infty} |x(t)| = 0$ (b) $\lim_{t\to\infty} |x(t)| = \infty$ (c) $\lim_{t\to\infty} |x(t)| = 2$ (d) $\lim_{t\to\infty} |x(t)| = 12$.
Ans. (a).

8. Let $a, b \in R$. Let $y = (y_1, y_2)'$ be a solution of the system of equations

$$y_1' = y_2, \ \ y_2' = ay_1 + by_2$$

Every solution $y(x) \to 0$ as $x \to \infty$ if **GATE(MA)-08**
A) $a < 0, b < 0,$ B) $a < 0, b > 0,$ C)$a > 0, b > 0$ D) $a > 0, b < 0$
Ans. A)

9. The system of ODE

$$\frac{dx}{dt} = (1+x^2)y,\ t \in \mathbb{R}$$
$$\frac{dy}{dt} = -(1+x^2)x,\ t \in \mathbb{R}$$
$$(x(0),\ y(0)) = (a,b)\mathbf{NET}(\mathbf{MS})(\mathbf{Dec}) - \mathbf{2014}$$

has a solution
(a) only if (a, b)=(0, 0) (b) for only $(a,b) =\in \mathbb{R}\times\mathbb{R}$
(c) such that $x^2(t)+y^2(t) = a^2+b^2$ for all $t \in \mathbb{R}$ (d) such that $x^2(t)+y^2(t) \to \infty$ as $t \to \infty$ if $a > 0$ and $b > 0$.
Ans. (b) and (c).

10. Let k be a real constant. The solution of the differential equations $\frac{dy}{dx} = 2y+z$ and $\frac{dz}{dx} = 3y$ satisfies the relation
(a) $y - z = ke^{3x}$ (b)$3y + z = ke^{3x}$
(c) $3y - z = ke^{3x}$ (d) $y + z = ke^{3x}$ [VU(CBCS)2018; JAM CA-2008]
Ans. (b)

11. If $y_1'(x) = 3y_1(x) + 4y_2(x)$ and $y_2'(x) = 4y_1(x) + 3y_2(x)$ then $y_1(x)$ is
(a) $c_1e^{-x} + c_2e^{7x}$ (b)$c_1e^{x} + c_2e^{7x}$
(c) $c_1e^{-x} + c_2e^{-7x}$ (d) $c_1e^{x} + c_2e^{-7x}$ [VU(CBCS)2018; JAM CA-2006]
Ans. (a)

12. Let $(x(t), y(t))$ satisfy for $t > 0$
$\frac{dx}{dt} = -x+y, \frac{dy}{dt} = -y,$ $x(0) = y(0) = 1.$
Then $(x(t))$ is equal to
1. $e^{-t} + ty(t)$ 2. $y(t)$ 3. $e^{-t}(1+t)$ 4. $-y(t)$ **[NET-DEC-2016]**
Ans: 1, 3.

13. The general solution of

$$y + \frac{dz}{dx} = 0$$
$$\frac{dy}{dx} - z = 0$$

is given by **GATE(MA)-05**
A) $\begin{cases} y = \alpha e^x + \beta e^{-x} \\ z = \alpha e^x \beta e^{-x} \end{cases}$ B) $\begin{cases} y = \alpha\cos x + \beta\sin x \\ z = \alpha\sin x - \beta\cos x \end{cases}$ C) $\begin{cases} y = \alpha\sin x - \beta\cos x \\ z = \alpha\cos x + \beta\sin x \end{cases}$ D) $\begin{cases} y = \alpha e^x \beta e^{-x} \\ z = \alpha e^x + \beta e^{-x} \end{cases}$
Ans. C)

14. The general solution of $\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$ is given by
A) $y = c_1, x^2+z^2 = c_2$ B) $y + x = c_1, x^2 + z = c_2$ C) $x = c_1, x + z^2 = c_2$ D) $y^2 + x = c_1, x + z = c_2$
Ans. A)

1.11 Review Exercises

1 Let A be a 3×3 matrix with real entries. If three solutions of the linear system of differential equations $\dot{x}(t) = Ax(t)$ are given by $\begin{pmatrix} e^t - e^{2t} \\ -e^t + e^{2t} \\ e^t + e^{2t} \end{pmatrix}\begin{pmatrix} -e^{2t} - e^{-t} \\ e^{2t} - e^{-t} \\ e^{2t} + e^{-t} \end{pmatrix}$ and $\begin{pmatrix} e^{-t} + 2e^t \\ e^{-t} - 2e^t \\ -e^{-t} + 2e^t \end{pmatrix}$. Then the sum of the diagonal entries of A is equal to $--$? **GATE(MA):2018**
Ans. 2.**Hint.** The independents solutions are e^{-t}, e^t, e^{2t}. So eigenvalues are -1, 1, 2. Hence the sum of the diagonal entries of A is equal to $\lambda_1 + \lambda_2 + \lambda_3 = -1 + 1 + 2 = 2$.

2 An n^{th} order ODE is equivalent to a system of n first order ODEs.

3 Define an initial value problem for a first order system. Reduce an initial value problem for an $n-$th order ODE to that of an equivalent n first order system.

4 Let $\mathbf{f}$ be a vector-valued function defined for $(t, \mathbf{x})$ in a set S with t real, $\mathbf{x} \in \Re^n$.
(a) Show that $\mathbf{f}$ is continuous at a point $(t_0,\ \mathbf{x_0})$ in S if and only if

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t_0, \mathbf{x_0})\| \to 0,$$

as $0 < |t - t_0| + \|x - x_0\| \to 0$.
(b) Show that $\mathbf{f}$ satisfies a Lipschitz condition in S if and only if each component of $\mathbf{f}$ satisfying a Lipschitz condition in S.

5 Show that $\mathbf{f}(x, \mathbf{y}) = (7x + 6y_1,\ y_1 + y_2)$ on $S:\ \{|x| < \infty,\ \ |y| < \infty\}$ satisfying a Lipschitz condition.

6 The system of n linear simultaneous ordinary differential equations is the form of

$$\begin{aligned}
\dot{x}_1(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\
\dot{x}_2(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\dot{x}_n(t) &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t)
\end{aligned} \tag{1.95}$$

and subject to the

$$x_i(t_0) = x_{i,0}, \ \text{ for } i = 1, 2, \cdots, n. \tag{1.96}$$

Suppose the coefficients a_{ij}, $(i, j = 1, 2, \cdots, n)$ and the functions f_i, $(i = 1, 2, \cdots, n)$ are continuous on the interval $[a,\ b]$. Then prove that the problem (1.95) with (1.96) has a unique solution $(x_1(t),\ x_2(t),\ \cdots,\ x_n(t))$ in $[a,\ b]$.

7 Show that all solutions with values in R^2 of the following system exist for all real t :

$$\begin{aligned}
x' &= a(t)\cos x + b(t)\sin y, \\
y' &= c(t)\sin x + d(t)\cos y,
\end{aligned}$$

where a, b, c, d are polynomials with real coefficients.
Hint. Apply the Theorem 1.4.

8 Consider the problem

$$\begin{aligned}
x' &= 3x + tz, \\
y' &= y + t^2 z \\
z' &= 2tx - y + e^t z.
\end{aligned}$$

Show that every initial value problem for this system has a unique solution which exists for all real t.

Hint. Apply the Theorem 1.4.

9 Let q be a real valued continuous function on $[-a, a]$. Then show that the initial value problem

$$\frac{d^2x}{dt^2} + \lambda^2 x = q(t)x,\ (\lambda \geq 0),\ x(0) = 0,\ \frac{x(0)}{dt} = 1$$

has a solution on $[-a, a]$.

Hint. Apply the Theorem 1.6.

10 Show that all real-valued solutions of the equation

$$\frac{d^2x}{dt^2} + \sin x = b(t)$$

where b is continuous for $-\infty < t < \infty$, exist for all real t.

11 Let $a_1, b_1, a_2, b_2 \in \Re$ Show that the condition $a_2b_1 > 0$ is sufficient but not necessary for the system. $\frac{dx}{dt} = a_1x + b_1y$ and $\frac{dy}{dt} = a_2x + b_2y$ to have two linearly independent solutions of the form $x = c_1e^{\lambda_1 t}$, $y = d_1e^{\lambda_1 t}$ and $x = c_2e^{\lambda_2 t}$, $y = d_2e^{\lambda_2 t}$ with $\lambda_1, \lambda_2, c_1, d_1c_2, d_2 \in \Re$ **JAM(MA)-2008**

12 Find the general solutions for the system

$$\begin{aligned} \frac{dy_1}{dt} &= 3y_1 + y_2 \\ \frac{dy_2}{dt} &= y_1 + 3y_2 \end{aligned}$$

Hint. The eigenvectors and eigenvalues of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

are $\lambda_1 = 4$ with

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and $\lambda_2 = 2$ with

$$\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Ans. The general solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}.$$

13 Find the solution for the system of differential equations

$$\frac{dy_1}{dt} = -3y_1 + 2y_2,\quad \frac{dy_2}{dt} = -2y_1 + 2y_2$$

Ans. The solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t}$$

14 Find the solution for the system

$$\frac{dy_1}{dt} = 3y_1 + y_2, \qquad \frac{dy_2}{dt} = y_1 + 3y_2$$

Ans. The solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}.$$

Solve the following system of simultaneous equations (D stands for $\frac{d}{dt}$).

15 $Dx - 7x + y = 0, \; Dy - 2x - 5y = 0.$
Ans. $x = e^{6t}(A\cos t + B\sin t), \; y = e^{6t}((A-B)\cos t + (A+B)\sin t).$

16 Find the general solutions for the system

$$\frac{dy_1}{dt} = 2y_1 - y_2, \qquad \frac{dy_2}{dt} = -4y_2$$

Ans. The general solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 6 \end{pmatrix} e^{-4t}.$$

17 Find the solution of

$$\frac{dy_1}{dt} = -y_1 - 2y_2, \qquad \frac{dy_2}{dt} = 2y_1 - y_2$$

Ans. The general solution is

$$\mathbf{y} = \begin{pmatrix} r\cos 2t \\ r\sin 2t \end{pmatrix} e^{-t}$$

18 Solve: $\frac{dx}{dt} = ny - mz, \; \frac{dy}{dt} = lz - nx, \; \frac{dz}{dt} = mx - ly$
Ans. $x^2 + y^2 + z^2 = c_1, \; lx^2 + my^2 + nz^2 = c_2, \; lx + my + nz = c_3.$

19 If $x(t)$ and $y(t)$ are the solutions of the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$ with initial condition $x(0) = 1$ and $y(0) = 1$ then find the value of $x(\frac{\pi}{2}) + y(\frac{\pi}{2})$. GATE(MA)-2017
Ans. 0.

20 $(D^2 - 4D + 4)x - y = 0, \; (D^2 + 4D + 4)y - 25x = 16e^t.$
Ans. $x = c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t - e^t, \; y = c_1e^{3t} + 25c_2e^{-3t} + (3c_3 - 3c_4)\cos t + (3c_4 + 4c_3)\sin t - e^t.$

21 Solve: $(D^2+1)x + (D+1)y = t, \; 2x + (D+1)y = 0$, given that $x(0) = y(0) = 0$ and $Dx(0) = -5.$ B.U(Hons.)-1999
Ans. $x = -2e^t + 2e^{-t} - t, \; y = 2(e^t - 2te^{-t} + t - 1)$

22 $Dx + 4x + 3y = t, \; Dy + 2x + 5y = e^t.$ [C.H,-1988]
Ans. $x = c_1e^{-2t} + c_2e^{-7t} - \frac{31}{196} + \frac{5}{14}t - \frac{1}{8}e^t, \; y = -\frac{2}{3}c_1e^{-2t} + c_2e^{-7t} - \frac{9}{98} - \frac{1}{7}t + \frac{5}{24}t.$

23 Obtain the G.S of the following system of differential equations: $Dx = x + 2y, \; Dy = 4x - 5y + e^{3t}.$ **JAM(MA)-2010**
Ans. $x = c_1e^{3t} + c_2e^{-3t} + \frac{1}{3}te^{3t}$ and $y = (c_1 + \frac{1}{6})e^{3t} - 2c_2e^{-3t} + \frac{1}{3}te^{3t}.$

24 Show that the integral of the equations $Dx + 2y = 0,\ Dy = x$ is given by $x^2 + y^2 + 2c = 0$. C.U(Hons.)-1989

25 Solve: $Dx + 5x + y = e^t,\ Dy - x + 3y = e^{2t}$. [C.H,-1993, K.U.-2002]
Ans. $x = (c_1 + c_2 t)e^{-4t} + \frac{4}{25}e^t - \frac{1}{36}e^{2t},\ y = -(c_1 + c_2 + c_2 t)e^{-4t} + \frac{1}{25}e^t - \frac{7}{36}e^{2t}$

26 Solve: $(D + 2)x + (D - 1)y = 3(t^2 - e^{-t}),\ (2D - 1)x - (D + 1)y = 3(2t - e^{-t})$. IAS 2003
Ans. $x = c_1 \cos t + c_2 \sin t + t^2 + e^{-t},\ y = \frac{1}{2}(3c_2 + c_1)\cos t + \frac{1}{2}(c_2 - 3c_1)\sin t + 2e^{-t} - t^2$.

27 Solve: $\frac{d^2x}{dt^2} - \frac{dy}{dt} = 2x + 2t,\ \frac{dx}{dt} + 4\frac{dy}{dt} = 3y$,
Ans. $x = (c_1 + c_2 t)e^t + c_3 e^{-\frac{3t}{2}} - t,\ y = (3c_2 - c_1 - c_2 t) - \frac{1}{6}c_3 e^{-\frac{3t}{2}} - \frac{1}{3}$.

28 Solve: $\frac{d^2y}{dt^2} - 16x = t,\ \frac{d^2x}{dt^2} - y = 1$, given that $y = 0,\ x = 0;\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = -\frac{1}{4}$ at $t = 0$.
Ans. $x = \frac{1}{16}(3e^{2t} - e^{-2t}) + \frac{9}{32}\sin 2t - \frac{1}{8}\cos 2t - \frac{1}{16}t,\ y = \frac{1}{4}(3e^{2t} - e^{-2t}) - \frac{9}{8}\sin 2t + \frac{1}{2}\cos 2t - 1$.

29 Solve the following initial value problem: $Dx + Dy - 2y = 2\cos t,\ Dx - Dy - 2x = 4\cos t$, given that $x = y = 0$ at $t = 0$.[CU(H) 2015]
Ans. $x = 2\cos t(e^t - 1) + \sin t,\ \ y = \sin t(1 - 2e^t)$.

30 Solve : $\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - x = e^t \cos t,\ \frac{d^2y}{dt^2} + 2\frac{dx}{dt} - y = e^t \sin t,$.
Ans. $x(t) = (c_1 + c_2 t)\cos t + (c_3 + c_4 t)\sin t + \frac{e^t}{25}(4\sin t - 3\cos t),\qquad y(t) = (c_3 + c_4 t)\cos t - (c_1 + c_2 t)\sin t - \frac{e^t}{25}(4\cos t + 3\sin t)$.

31 Solve: $tDx + 2(x - y) = t,\ \ tDy + x + 5y = t^2$.
Ans. $x(t) = c_1 t^{-3} + c_2 t^{-4} + \frac{3t}{10} + \frac{t^2}{15},\ y(t) = -\frac{c_1 t^{-3}}{2} - c_2 t^{-4} - \frac{t}{20} + \frac{2t^2}{15}$.

32 Find all solutions of the system $\dot{x} = A(t)x + f(t)$ with initial conditions $x(0) = [0,\ 1,\ -1]^T$ where $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ te^{-t} & te^{-t} & 1 \end{bmatrix}$ and $f(t) = \begin{bmatrix} e^t \\ 0 \\ 1 \end{bmatrix}$.

Ans.
$$x_1(t) = (\frac{3}{4} + \frac{1}{2}t)e^t - \frac{3}{4}e^{-t}$$
$$x_2(t) = (\frac{1}{4} + \frac{1}{2}t)e^t + \frac{3}{4}e^{-t}$$
$$x_3(t) = 3e^t - t^2 - 3t - 4.$$

33 Find the general solution of $\frac{dx}{cy-bz} = \frac{dy}{az-cx} = \frac{dz}{bx-ay}$ [**Ans.** $\phi(ax + by + cz, x^2 + y^2 + z^2) = 0$]

34 Find the general solution of $\frac{xdx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2}$ [**Ans.** $\phi(x^2 - z^2, x^3 - y^3) = 0$]

35 Find the general solution of $\frac{dx}{2xz} = \frac{dy}{2yz} = \frac{dz}{z^2-x^2-y^2}$ [**Ans.** $\phi(\frac{x}{y}, \frac{z^2+(x+y)y}{y}) = 0$]

36 Find the general solution of $\frac{dx}{x(x+y)+az} = \frac{dy}{y(x+y)-az} = \frac{dz}{z(x+y)}$ [**Ans.** $x^2 - y^2 - 2az = \phi(\frac{x+y}{z})$]

37 Find the general solution of $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$. [**BU(H) 99**]
Ans. $\phi(\frac{x^2+y^2+z^2}{z}, \frac{y}{z}) = 0$.

38 Find the general solution of $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$.
Ans. $z^{\sqrt{2}} \cot\left(\frac{x+y+\frac{\pi}{4}}{2}\right) = \phi\left(\frac{\cos(x+y)+\sin(x+y)}{e^{x-y}}\right)$.

39 Find the general solution of $\frac{dx}{x(2y^4-z^4)} = \frac{dy}{y(z^4-2x^4)} = \frac{dz}{z(x^4-y^4)}$.
Ans. $x^4 + y^4 + z^4 = \phi(xyz^2)$.

40 Find a fundamental matrix for the system

$$\frac{dX(t)}{dt} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X(t).$$

Ans. $\Phi(t) = \begin{pmatrix} e^t & 0 & te^t \\ 0 & 0 & e^t \\ 0 & e^t & 0 \end{pmatrix}$.

41 Find a fundamental matrix for the system

$$\frac{dy_1}{dt} = y_1 + y_2, \qquad \frac{dy_2}{dt} = y_1 + y_3, \qquad \frac{dy_3}{dt} = y_3.$$

Find also the solution.

Ans. $\Phi(t) = \begin{pmatrix} e^t & te^t & \frac{t^2e^t}{2} \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}$.

The solution is $y_1(t) = Ae^t + Bte^t + C\frac{t^2e^t}{2},\ y_2(t) = Ae^t + Bte^t,\ y_3(t) = Ae^t$.

42 Find the fundamental matric and the solution $x(t)$ such that $x(0) = [3\ \ 1]^T$ for the system.

$$\frac{dx_1}{dt} = x_1 - 2e^{-t}x_2, \qquad \frac{dx_2}{dt} = e^t - x_2.$$

Ans. $\Phi(t) = \begin{pmatrix} 2 & e^{-t} \\ e^t & 1 \end{pmatrix}$ and $x(t) = \begin{pmatrix} 4 - e^{-t} \\ 2e^{-t} - 1 \end{pmatrix}$

43 Solve : $\frac{dx}{y^2+yz+z^2} = \frac{dy}{z^2+zx+x^2} = \frac{dz}{x^2+xy+y^2}$.

Ans. $(xy + yz + zx)^2 - (x^2 + y^2 + z^2)^2 = \phi(\frac{y-x}{z-y})$.

44 Solve : $\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)}$. **[BU(H) 2015]**

Ans. $(x - y)^{-1} - (x + y)^{-1} = \phi(\frac{(x+y)}{z})$.

45 Solve : $\frac{dx}{x(x^2+3y^2)} = \frac{dy}{y(y^2+3x^2)} = \frac{dz}{2z(x^2+y^2)}$.

Ans. $(x + y)^{-2} - (x - y)^{-2} = \phi(\frac{(xy)}{z^2})$.

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