Integration

$$\frac{d}{dx}[f(x)] = \phi(x)$$

$$d\{f(x)\} = \phi(x)dx$$

We then say that the integrals of $\phi(x)$ w.r.t x is $f(x)$ and write, $\int \phi(x)dx = f(x)$

Standard Formula :-

a) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ (except when $n = -1$).
where c = constant of integration.

b) $\int x^{-1} dx = \log|x| + c$ $(x>0)$

c) $\int 1 dx = x + c$

d) $\int e^x dx = e^x + c$

e) $\int e^{mx} dx = \frac{e^{mx}}{m} + c$

f) $\int a^x dx = \frac{a^x}{\log a} + c$

g) $\int a^{mx} dx = \frac{a^{mx}}{m\log a} + c$

Theorem

1. $\int a f(x) dx = a \int f(x) dx$.

2. $\int \{f(x) \pm \phi(x)\} dx = \int f(x) dx \pm \int \phi(x) dx$

Two Important Result :-

1. $\int \frac{dx}{x \pm a} = \log|x \pm a| + c$

2. $\int \frac{dx}{ax \pm b} = \frac{1}{a} \log|ax \pm b| + c$

Example

1. $\int x^7 dx$

$= \frac{x^{7+1}}{7+1} + c$

$= \frac{x^8}{8} + c$

2. $\int \frac{dx}{x^{12}}$

$= \int x^{-12} dx$

$= \frac{x^{-12+1}}{-12+1} + c$

$= -\frac{1}{11x^{11}} + c$

3. $\int e^{3x} dx$

$= \frac{e^{3x}}{3} + c$

$= \frac{1}{3} e^{3x} + c$

4. $\int 2^x dx$

$= \frac{2^x}{\log 2} + c$

5. $\int \frac{(x+1)^2}{\sqrt{x}} dx$

$= \int \frac{x^2+2x+1}{\sqrt{x}} dx$

$= \int (x^{3/2} + 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}) dx$

$= \int x^{3/2} dx + 2\int x^{1/2} dx + \int x^{-\frac{1}{2}} dx$

$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{2}{5} x^{5/2} + \frac{4}{3} x^{3/2} + 2x^{1/2} + c$

6. The slope of a curve at (x, y) is $9x$. If it passes through the origin, show that its equation is $9x^2 = 2y$

Soln. The slope of the curve

$\frac{dy}{dx} = 9x$

or $dy = 9x\,dx$

or $\int dy = \int 9x\,dx$

or $y = 9\frac{x^2}{2} + c$

$\therefore 9x^2 = 2y$

7. Find a function whose derivative is $\frac{x^2}{x+1}$

Soln. Let the required function be y.

Then $\frac{dy}{dx} = \frac{x^2}{x+1}$

or $dy = \frac{x^2}{x+1}\,dx$

or $\int dy = \int \frac{x^2}{x+1}\,dx$.

or $\int dy = \int \frac{x^2-1+1}{x+1}\,dx$.

or $y = \int \frac{(x+1)(x-1)}{x+1}\,dx + \int \frac{1}{x+1}\,dx$

$= \int x\,dx - \int dx + \int \frac{dx}{x+1}$

$= \frac{x^2}{2} - x + \log|x+1| + c$ where c is a constant of integration.

8. Evaluate:

$\int \frac{x^3+1}{x+1}\,dx$

$= \int \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}}\,dx$

$= \int x^2\,dx - \int x\,dx + \int dx$

$= \frac{x^3}{3} - \frac{x^2}{2} + x + c$

9. $\int \frac{(e^x+1)^3}{e^{2x}}$

$= \int \frac{e^{3x}+3e^{2x}+3e^x+1}{e^{2x}}\,dx$

$= \int (e^x + 3 + 3e^{-x} + e^{-2x})\,dx$

$= \int e^x\,dx + 3\int dx + 3\int e^{-x}\,dx + \int e^{-2x}\,dx$

$= e^x + 3x - 3e^{-x} - \frac{1}{2}e^{-2x} + c$

$= e^x + 3x - \frac{3}{e^x} - \frac{1}{2e^{2x}} + c$

10. $\int \frac{e^{2x}+e^{4x}}{e^{x}+e^{-x}} dx$

$= \int \frac{e^{3x}(e^{-x}+e^{x})}{e^{x}+e^{-x}} dx$

$= \int e^{3x} dx$

$= \frac{1}{3} e^{3x} + c$

11. $\int (e^{3\log x} - e^{x\log 3}) dx$

$= \int e^{3\log x} dx - \int e^{x\log 3} dx$

$= \int e^{\log x^3} dx - \int e^{\log 3^x} dx$

$= \int x^3 dx - \int 3^x dx$

$= \frac{1}{4} x^4 - \frac{3^x}{\log 3} + c$

12. $\int \frac{8^{1+x} - 4^{1-x}}{2^x} dx$

$= \int \frac{2^{3+3x} - 2^{2-2x}}{2^x} dx$

$= \int (2^{3+3x-x} - 2^{2-2x-x}) dx$

$= \int (2^{3+2x} - 2^{2-3x}) dx$

$= \int 2^3 \cdot 2^{2x} dx - \int 2^2 \cdot 2^{-3x} dx$

$= 2^3 \int 2^{2x} dx - 2^2 \int 2^{-3x} dx$

$= 8 \cdot \frac{2^{2x}}{2\log 2} - 4 \cdot \frac{2^{-3x}}{-3\log 2} + c$

$= \frac{4 \cdot 2^{2x}}{\log 2} + \frac{4}{3} \cdot \frac{2^{-3x}}{\log 2} + c$

$= \frac{4}{\log 2} (2^{2x} + \frac{1}{3} 2^{-3x}) + c$